PRINCIPLES
OF
ELECTRICAL ENGINEERING

BY

HAROLD PENDER, PH.D.

PROFESSOR OF THEORETICAL AND APPLIED ELECTRICITY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
MEM. A. I. E. E.

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PREFACE

This book is an exposition of the physical principles upon which the art of electrical engineering is based, together with a discussion of the application of these principles in some of the simpler forms of electric apparatus and machinery. The first four chapters of the book may be looked upon as an introduction to the detailed study of continuous current machinery, the last five to the detailed study of alternating current machinery and the transmission and distribution of energy by alternating currents. The text is the development of a course of lectures given by the author to the junior class in electrical engineering at the Massachusetts Institute of Technology. The time allotted to this course, exclusive of home preparation, is fifty-five hours of lectures, twenty-three hours of recitations, and twenty-three hours for the solution of problems under the supervision of an instructor.

A preliminary edition of the book was published in 1910, primarily for the use of the students at the Institute, and this preliminary edition was preceded by mimeographed notes covering practically the same ground. The present edition differs from the preliminary edition chiefly in the addition, at the end of each chapter, of a summary of the important definitions and principles developed in that chapter and of a list of problems, with answers, illustrative of these principles. Certain sections of the text have also been rewritten, particularly the sections on energy, inductance, and capacity; the conception of linkages between the electric and magnetic circuits has also been more fully developed. The typographical errors of the preliminary edition have been corrected and additional steps have been inserted in some of the mathematical deductions.

There will undoubtedly be those who, after an examination of the text, will pronounce it more of a treatise on physics than on electrical engineering. But electrical engineering is primarily the application of physics, and of necessity the principles of electrical engineering are the principles of physics. This fact is

too frequently overlooked, and the student is rushed into the study of electric machinery and other advanced subjects with but the vaguest conception of the physical principles upon which the operation of electric apparatus is based. In the author's opinion, a clear conception of the principles of physics and the ability to apply these principles in co-ordinating the experimental facts of physics, both qualitatively and quantitatively, is absolutely essential before one can get a clear understanding of the more complicated reactions that take place in electric machinery and transmission circuits. It is with the hope that others may find this text useful in filling the gap between elementary physics and applied electricity that the author offers it to the public.

The method of treatment adopted throughout is to describe first certain simple and typical experiments which illustrate a given principle, second, to state the principle in an exact manner in its general form, and then explain the application of the principle in one or more practical cases. The problems given at the end of each chapter serve as a further illustration of the principles developed. The attempt has also been made to make each section lead naturally into the next and to show how the various phenomena of electricity and magnetism are interrelated. The analogy between the flow of electricity and hydraulics is brought out repeatedly, and emphasis is laid upon the similarity of magnetic and electrostatic phenomena. In the discussion of alternating currents sine functions are used until the meaning of the various terms, such as effective value, phase difference, etc., has been made clear; the vector method is then introduced, and finally the symbolic method is developed.

The calculus is employed from the very beginning of the book. The calculus is taught the students of electrical engineering in practically every technical school in the country. The author has adopted the common-sense view that since the student is provided with so useful a tool he should learn how to use it, particularly as this tool is one of the greatest labor-saving devices ever invented. In every case, however, the physical meaning of the formulas developed has been clearly stated.

The problems at the end of each chapter are of two kinds, which for convenience may be designated as practical and theoretical. To fix principles in the student's mind problems may be devised which, though seldom met with in practice, are

much more effective than purely practical ones. Practical problems, however, should not be neglected, for the student should also gain facility in making the simple calculations of ordinary practice. In addition, many of the problems of the latter class have been selected to bring out the principles of certain special phenomena and methods of practice which are not treated in detail in the text.

To those who may use the book in the class-room the following suggestions are offered. When the time available is limited, the articles printed with close spacing, for example, Article 42, may be omitted. The student should be made to understand that the summaries at the end of each chapter are not to be memorized, but are given merely as a bird's-eye view of the chapter. The solution of as many problems as possible in conjunction with the study of the text will also enable the student to test his understanding of the latter; it is with this object in view that the answers to the problems have been given. The student should be required to follow closely in all written work the recommendations of the American Institute of Electrical Engineers, given in Appendix A.

The author takes this opportunity of expressing his indebtedness to Messrs. Cary T. Hutchinson, W. A. Del Mar, and H. S. Osborne and to the various members of the Electrical Engineering Department of the Massachusetts Institute of Technology for many valuable suggestions and criticisms. The majority of the problems in this edition were prepared by Mr. R. G. Hudson, to whom the author is also indebted for his assistance in the reading of proof.

HAROLD PENDER

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I

FUNDAMENTAL IDEAS AND UNITS

1. Introduction. — The fundamental conceptions with which we have to start are the ideas of space, time and matter. We notice all about us that matter is changing its position in space, or moving, and that each motion requires a certain interval of time. By measuring the amount of matter involved, the amount of motion that takes place and the time required, scientists have found that these quantities are invariably related in definite quantitative ways; in other words, that every change in the motion of matter takes place in accordance with definite laws.

Many of the fundamental conceptions used in scientific work are based on assumptions which are incapable of proof, but which, on account of their simplicity and plausibility, we accept as true. For example, we accept as true that the interval of time required for a given change in the position of a given portion of matter will always be the same, provided the conditions under which the change takes place are exactly duplicated; again, we accept as true that whenever the velocity of a particle of matter changes, this change is due to the influence of some other particle or particles of matter or to some agent associated with the particle or particles. Such assumptions have been called "Articles of Scientific Faith."

In order to express the laws of nature in a quantitative manner, it is necessary to define clearly 1. What shall be taken as a measure of each quantity, 2. What is meant by equal amounts of this quantity, and 3. What shall be taken as the unit of this quantity.

2. Length. — Two straight lines which can be superimposed one upon the other, so that the ends of the two lines exactly coincide, are said to be equal in length. The distances between any two pairs of points AB and $A'B'$ respectively are said to be equal when the straight lines drawn between A and B and between A' and B' are equal. If we choose the distance between any two arbitrary points as a unit or standard of length, then any other

length may be expressed as the number of these equal unit lengths into which the line between the two given points may be divided.

The unit or standard of length employed in all scientific work is the *centimeter*, abbreviated *cm.*; it is the $\frac{1}{100}$th portion of the length of a certain platinum-iridium bar known as the International Meter and preserved at the International Bureau of Weights and Measures near Paris; the length of the bar is measured when it is at the temperature of melting ice, *i.e.*, at 0° centigrade. Some of the other common units of length employed in scientific and engineering work are related to one another as follows:

1 meter	=100 centimeters
1 millimeter	=0.1 centimeter
1 kilometer	=1000 meters
1 inch	=2.5400 centimeters
1 mil	=0.001 inch
1 foot	=30.480 centimeters
1 yard	=91.440 centimeters
1 mile	=5280 feet
1 mile	=1.6093 kilometers
1 mile	=1609.3 meters

3. **Surface.** — The unit of surface is the area of a square each side of which is one unit in length; some of the common units of surface are related to one another as follows:

1 square inch	=6.4516 square centimeters
1 circular mil	$=0.78540\times10^{-6}$ square inch
1 circular mil	=0.00050671 square millimeter
1 square foot	=929.03 square centimeters
1 square yard	=8361.3 square centimeters
1 acre	=43,560 square feet
1 acre	=4046.9 square meters
1 square mile	=27,878,400 square feet
1 square mile	=640 acres
1 square mile	=2.5900 square kilometers

4. **Volume.** — The unit of volume is the volume of a cube each edge of which is one unit in length. Some of the common units of volume are related to one another as follows:

1 liter	=1000 cubic centimeters
1 cubic inch	=16.387 cubic centimeters
1 cubic foot	=28,317 cubic centimeters
1 cubic foot	=1728 cubic inches

1 cubic foot	=7.4805 gallons (Liquid; U. S.)
1 cubic yard	=0.76456 cubic meter
1 quart (Liquid; U. S.)	=0.94636 liter
1 gallon (Liquid; U. S.)	=231 cubic inches
1 gallon (Liquid; U. S.)	=3.7854 liters

5. Angle. — Let AB and AC in Fig. 1 be two straight lines intersecting in the point A. With A as a center and any distance as a radius, draw a circle about A in the plane ABC. Let B' and C' be the points where this circle cuts the lines AB and BC respectively. Then the ratio of the arc $B'C'$ to the radius AB' or AC' is called the angle between the lines AB and BC; that is,

$$\text{Angle } CAB = \frac{arc\ B'C'}{AB'}. \tag{1}$$

This ratio is independent of the length AB', since the arc is proportional to the radius. The unit angle as thus defined is called the "radian"; that is, a radian is the angle subtended by an arc which is equal to the radius. Angles are also expressed in terms of another arbitrary unit called the "degree." One degree is the angle subtended by $\frac{1}{360}$th part of the arc of a circle. Since the total length of a circumference is equal to $2\pi \times$(radius), the total plane angle about a point is equal to 2π radians. Also, from the definition of the degree, the total plane angle about a point is 360 degrees. Hence

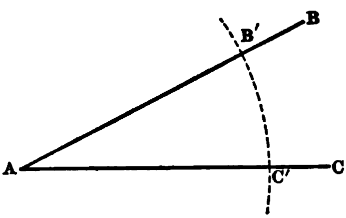


Fig. 1.

$$1 \text{ radian} = 57.296 \text{ degrees.}$$

The angle between two planes which intersect in a line MN is defined as the angle between the two lines in which these two planes intersect a third plane drawn perpendicular to MN.

The angle made by a given line with any other line which it does not intersect is defined as the angle between the given line and a line drawn through any point in this line parallel to the second line.

6. Time. — We accept as an article of scientific faith that the time required for a given change in the position of a given portion of matter will always be the same provided all the other conditions

under which the displacement occurs are exactly duplicated. The simplest motion of this kind is that of a pendulum suspended at a given point with reference to the earth and oscillating so that any given point of the pendulum passes over the same distance during each oscillation. We may then take as the numerical measure of any interval of time the number of vibrations made in this interval by such a pendulum. The unit or standard of time ordinarily adopted in scientific work is the time required for one oscillation of a pendulum, which, when kept under absolutely constant conditions, would make 86,400 oscillations in a mean solar day; this unit is called the *second*. The solar day is the interval of time between two successive transits of the sun across the meridian of the earth at the point of observation; this interval varies in length at different times during the year but the average length of the interval for one year is constant as far as we know. Some of the common units of time are related to one another as follows:

1 hour	=3600 seconds
1 day	=86,400 seconds
1 civil or calendar year	=8760 hours
1 civil or calendar year	=31,536,000 seconds
1 solar year	=365.2422 days
1 solar year	=31,556,926 seconds
1 leap year	=8784 hours

7. Displacement. — Let a point which had a position P at any instant have at some later instant a position Q. The straight line drawn from P to Q is called the *linear displacement* of the point. Let AB be any other straight line in space and imagine a plane drawn through P and the line AB and another plane through Q and the line AB; the angle between the planes PAB and QAB is called the *angular displacement* of the point *about the axis* AB.

8. Vectors. — The line PQ representing the linear displacement of a point has both length and direction; the length of the line may be represented by a number and its direction by the angles made by PQ with any three arbitrarily chosen axes of co-ordinates. In the majority of problems that arise in engineering work the various points of a body move in parallel planes; in this case the direction of the linear displacement PQ of any point can be expressed numerically by the angle made by the

line PQ with an arbitrary line of reference fixed in any one of these planes. For example, in Fig. 2 let the plane of the paper be the plane in which the point moves, let OX be a line fixed in this plane and let PQ be the linear displacement of the point. Draw a line through P parallel to OX and let θ be the angle between this line and PQ; then both the amount and direction of the linear displacement of the point are completely determined when we know the magnitude of the length PQ and its direction. A quantity of this kind which requires for its complete representation a magnitude and a direction is called a *vector quantity* and the line representing such a quantity is called a *vector*. A quantity which has magnitude only, but not direction, such as time, mass, etc., is called a *scalar quantity.*

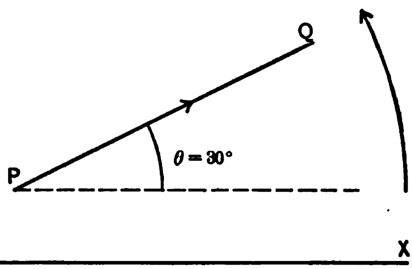


Fig. 2.

When a point moves from P to Q and then back from Q to P, the final displacement of the point is zero; this may be expressed mathematically by the formula $PQ+QP=0$, or $PQ=-QP$. That is, the vector PQ is equal but opposite to the vector QP; the vector QP is said to be in the opposite *sense* to the vector PQ. It is therefore necessary in dealing with vectors to specify definitely the sense of the vector; this may be done by writing the letters representing the ends of the vector in the order such that motion from the first to

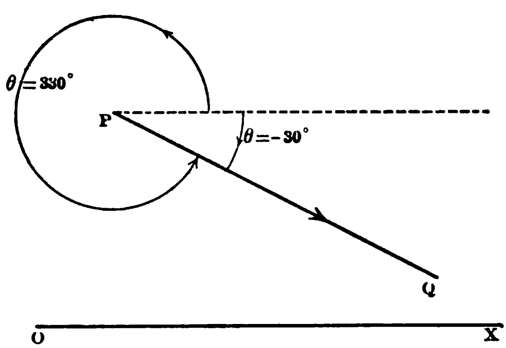


Fig. 3.

the second is in the positive sense of the vector, or by placing on the line representing the vector an arrow pointing in the positive sense of the vector. In dealing with vectors which lie in the same or in parallel planes (*i.e.*, *co-planar vectors*), it is usual to select an arbitrary line drawn from an arbitrary point as the axis of reference and to take as the positive sense of each vector its sense *away from* this arbitrary point of origin. The direction of each vector may then be expressed by the angle measured around to the vector in the counter-clockwise direction from the line of reference. For example, in Fig. 2, the vector PQ makes an angle of 30° with the line of reference OX. In Fig. 3 the vector PQ makes an angle of 330° with OX, which is equivalent to an angle of $-30°$ with OX.

9. Composition of Vectors. — Since by definition the linear displacement of a point when it moves from a position P to a position Q is the straight line PQ, this displacement is independent of the actual path over which the point moves from P to Q. For example, in Fig. 4, the point may move in a straight line from P to any other point B, then in a straight line to a point C, and finally to the point Q. The lines PB, BC and CQ are called the *components* of the vector PQ, and PQ is called the *resultant* of the vectors PB, BC, CQ. PQ may therefore be considered to be made up of any number of component vectors, provided that when all of these components are placed end to end they form a continuous path from P to Q such that a point moving from P to Q over this path moves in the positive sense of each component successively. Similarly the resultant of any number of vectors PB, BC, CQ, etc., is found by placing the lines representing these vectors end to end in such a manner that a point moving along the path formed by these lines always moves in the positive sense of the vectors; the line drawn from the beginning of the first vector to the end of the

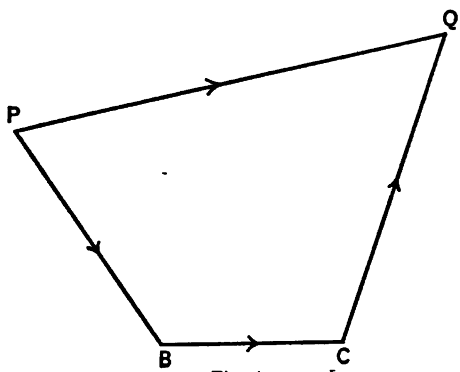


Fig. 4.

last of the series is then the resultant. The above facts may be represented by a formula thus: $PQ = \overline{PB + BC + CQ}$ where the line over the second term indicates that the lines PB, BC, and CQ must be "added" in the manner just described. Addition of this sort is usually called geometric or vector addition; the line over the second term then indicates that PB, BC and CQ must be added geometrically or vectorially.

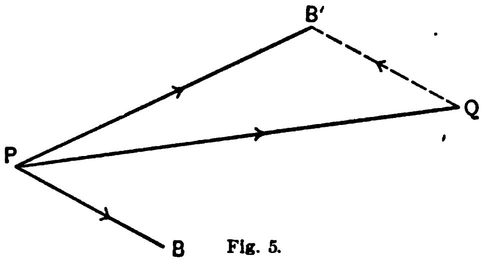


Fig. 5.

Similarly, we may subtract a vector PB from any other vector PQ by adding to PQ a vector QB' equal to PB and in the opposite or negative sense. This may be represented by a formula thus: $PB' = \overline{PQ - PB}$. (See Fig. 5.)

The addition of two or more co-planar vectors may also be expressed analytically. For example, let it be required to find the resultant R of the three vectors A, B, and C, Fig. 6. Choose any arbitrary line OX as a line of reference and let θ_r, θ_a, θ_b and θ_c be the angles made by R, A, B, and C respectively with the line OX, and call these angles *positive* when measured in the *counter-clockwise* direction around from OX and *negative* when measured in the *clockwise* direction. Then the component of R parallel to OX is

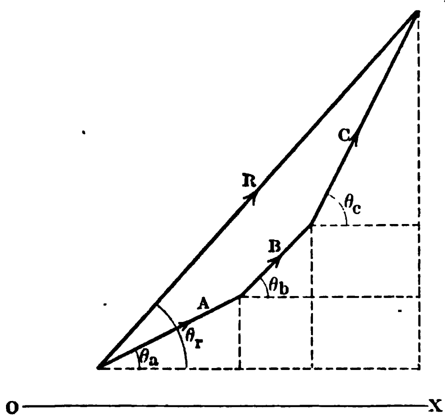


Fig. 6.

$$R \cos \theta_r = A \cos \theta_a + B \cos \theta_b + C \cos \theta_c$$

and the component of R which makes an angle of 90° with OX is

$$R \sin \theta_r = A \sin \theta_a + B \sin \theta_b + C \sin \theta_c$$

Hence the length of R is

$$R = \sqrt{(A \cos \theta_a + B \cos \theta_b + C \cos \theta_c)^2 + (A \sin \theta_a + B \sin \theta_b + C \sin \theta_c)^2} \quad (2a)$$

and the angle which R makes with OX is θ_r, where

$$tan\ \theta_r = \frac{A\ sin\ \theta_a + B\ sin\ \theta_b + C\ sin\ \theta_c}{A\ cos\ \theta_a + B\ cos\ \theta_b + C\ cos\ \theta_c} \qquad (2b)$$

For example, let

$A=3$ $\qquad \theta_a=30°$

$B=2$ $\qquad \theta_b=45°$

$C=5$ $\qquad \theta_c=60°$

Then

$$R=\sqrt{(3\times0.866+2\times0.707+5\times0.5)^2+(3\times0.5+2\times0.707+5\times0.866)^2}=9.74$$

$$tan\ \theta_r = \frac{7.244}{6.512}=1.1124$$

$$\theta_r=48.05°$$

We shall see later on that many of the quantities met with in engineering problems, such as velocity, force, moments, electric current, etc., can be represented by vectors. In certain cases it will be seen that it will be unnecessary to specify the location of the vector, but that any two vectors which are equal in length and are parallel may be considered equivalent; in other cases we shall find that we may consider two equal and parallel vectors as equivalent only when they lie in the same plane, or in the same line, or it may be that the vector cannot be considered equivalent to any other vector. When the location of a vector is thus limited it is said to be localized in a plane, on a line or at a point, as the case may be. The above laws for the composition and resolution of vectors apply to localized vectors only in case the vectors, or vectors equivalent to them, meet in a point.

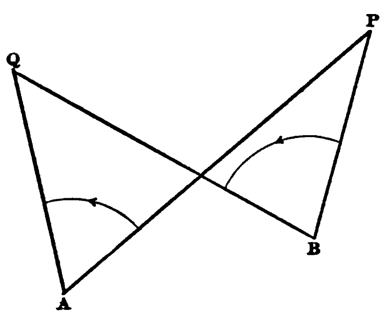


Fig. 7.

The angular displacement of a point about any axis is a quantity which requires for its representation a vector localized in a line. For example, let P and Q, lying in the plane of the paper, be the initial and final positions of the point; let the axis of rotation be a line drawn perpendicular to this plane at A. By definition, the angular displacement of the point about this

axis is then the angle PAQ. A line drawn in the axis through A having a length equal numerically to the angle PAQ will serve to represent both the axis of rotation and the numerical value of the angular displacement. To represent the sense of the rotation, *i.e.*, whether from P to Q or from Q to P, it is customary to choose arbitrarily the positive sense of the axis, and then call the rotation positive when the motion of the point is in a clockwise or right-handed direction when viewed by a person looking in the sense of this line. In the case illustrated, if we choose the positive sense of the axis to be toward the reader the line representing the angular displacement from P to Q will be drawn upward perpendicular to the plane of the paper; the line representing the displacement from Q to P will be drawn downward. The angular displacement of a point about any axis is therefore completely defined by a line having (1) a definite length, (2) a definite direction, and (3) localized in the axis of rotation. We may also represent the displacement of the point from the position P to the position Q by an angular displacement about any other axis, such as an axis through B parallel to the axis through A, but the vector representing this angular displacement about the axis through B will not be equal to the vector representing the angular displacement about the axis through A, since the angles PAQ and PBQ are not equal. Again, two equal and parallel vectors do not represent equivalent angular displacements unless these two vectors lie in the same line.

10. Velocity. — Let, in any small interval of time dt, a point P be displaced a distance dl with respect to any other point O; then the limiting value at any instant of the ratio $\frac{dl}{dt}$, when dt is taken extremely small, is called the *linear velocity* of the point P at that instant relative to the point O. Representing linear velocity by v, we have

$$v = \frac{dl}{dt}. \tag{3}$$

When the point moves in a straight line over equal distances in equal small intervals of time its linear velocity is said to be *uniform;* in this case the linear velocity of the point may be defined as the linear displacement of the point in unit time. Since linear displacement is a vector quantity, *i.e.*, has both magnitude and direction, and time is a scalar quantity, linear velocity is also a

vector quantity; for a vector quantity divided by a scalar is a vector quantity. Therefore a change either in the *magnitude* or in the *direction* of the linear velocity constitutes a *change* in the linear velocity. The magnitude of the linear velocity of a point is frequently called the linear *speed* of the point; that is, the linear speed of a point is simply a number expressing the distance over which the point moves in unit time; linear speed is therefore a scalar quantity. For example, a point moving in a circle in such a manner that in equal small intervals of time it passes over equal distances measured along the circumference of the circle, is said to be moving with a constant speed; its velocity, however, is changing at every instant, since the direction of motion is continually changing.

Linear speeds may be expressed in various units, such as the number of centimeters per second, feet per second, miles per hour, etc. The more common units are related as follows:

1 kilometer per hour	=0.91134 foot per second
1 kilometer per hour	=54.681 feet per minute
1 mile per hour	=0.44704 meter per second
1 mile per hour	=1.46667 feet per second
1 mile per hour	=88 feet per minute

Angular velocity is defined in exactly the same way as linear velocity, *i.e.*, the angular velocity of a point about any axis is

$$\omega=\frac{d\,\theta}{dt} \tag{4}$$

where $d\theta$ is the angular displacement of the point about that axis in the small interval of time dt. Angular velocity is represented by a localized vector in the same way that angular displacement is represented by a localized vector. The term angular speed is used to express the magnitude of the angular velocity in the same way that linear speed is used to express the magnitude of linear velocity. Angular speeds may be expressed in degrees per unit time, radians per unit time or revolutions per unit time. The more common units are related as follows:

1 radian per second	=57.296 degrees per second
1 radian per second	=0.159155 revolution per second

11. **Acceleration.** — The rate of increase of linear velocity with respect to time is called the *linear acceleration; i.e.*, when the linear

velocity of a point increases by a small amount dv in a small interval of time dt, then the linear acceleration is

$$a=\frac{dv}{dt}. \tag{5}$$

Note that dv is the difference between the two vectors representing the linear velocities at the beginning and end of the small interval of time dt, and that in general the direction of the vector representing this difference will have no fixed relation to the direction of the vectors representing these two velocities. However, when the direction of the line of motion does not change, *i.e.*, when these two velocities are in the same direction, the acceleration will likewise be in the same or the opposite direction to the velocity. In this case the linear acceleration is equal to the second derivative of the displacement, that is,

$$a=\frac{d^2l}{dt^2}, \tag{6}$$

where dl is the linear displacement in time dt. On the other hand, when the speed remains constant (*i.e.*, only the direction of the velocity changes) it can be readily shown that this vector difference, and therefore the direction of the acceleration a, is at every instant perpendicular to the path of motion and is toward the center of curvature of this path, and is equal to the square of the linear speed s divided by the radius of curvature r of this path; that is,

$$a=\frac{s^2}{r}. \tag{7}$$

The commonest linear acceleration with which we have to deal is the acceleration of falling bodies, or, as it is commonly called, the acceleration "due to gravity." This acceleration is constant for all kinds, shapes and sizes of bodies falling in a vacuum, for any given place on the earth's surface, but varies slightly with the latitude and with the altitude of the point of observation. At mean sea level and 45° latitude its value, as determined by Helmert in 1884, is 980.5966 centimeters per second. This value is sometimes used as the unit of acceleration; it is called the acceleration of gravity and is represented by the symbol g. Other common units of linear acceleration are related as follows:

1 kilometer per hour per second
=27.778 centimeters per second per second
=0.62137 mile per hour per second
=0.028327 gravity

1 mile per hour per second
=44.704 centimeters per second per second
=1.46667 feet per second per second
=0.045589 gravity
=3600 miles per hour per hour

Gravity =980.5966 centimeters per second per second
Gravity =32.172 feet per second per second

The variation of gravity with altitude and location is very slight, and the approximate values 981 centimeters per second per second and 32.2 feet per second per second are as a rule sufficiently accurate for engineering work, independent of altitude or location.

The *angular acceleration* of a point is similarly defined as the rate of change of its angular velocity. In case the point is rotating in a circle about a fixed axis, its angular acceleration may be defined as the rate of change of its angular speed about this axis. If ω is the angular speed then the angular acceleration is

$$a=\frac{d\omega}{dt}. \tag{8}$$

Since $\omega=\frac{d\theta}{dt}$, where $d\theta$ is the angular displacement in time dt, we also have, under the same conditions, that

$$a=\frac{d^2\theta}{dt^2}. \tag{9}$$

12. Mass. — The quantity of matter in a body or its *mass* can be defined only in terms of some effect produced on the body by some other body or bodies exterior to it. It has been found by experiment that two bodies, which appear to our senses to be identical in every respect, will exactly counter-balance each other when suspended one from each end of an equal-armed balance in a vacuum. We may, then, go a step further and define the mass of any two bodies as equal irrespective of their volume, shape or chemical composition, if, when they are suspended simultaneously in a vacuum, one from each end of an equal-armed balance, there is no tipping of the beam of the balance from its original position. This criterion for the equality of two masses holds only in case the

bodies and the balance are neither electrically charged nor magnetised, and both bodies are supported at the same distance from the earth, and the equilibrium of the balance is not affected by the presence of any other bodies (except the earth) in the vicinity.

This is an entirely arbitrary definition, but it has been found that mass as thus defined is a fundamental property of matter. Any arbitrary portion of matter may be taken as the unit of mass; the mass of any given portion of matter may then be expressed as the number of such equal units, which taken together, and suspended from one arm of an equal-armed balance in a vacuum, will just counter-balance the given body suspended from the other arm. Note that mass is a scalar quantity.

The unit or standard of mass adopted in all scientific work is the *gram*, abbreviated *g;* it is the $\frac{1}{1000}$th portion of a certain platinum-iridium cylinder, known as the International Kilogram and preserved at the International Bureau of Weights and Measures near Paris. Some of the common units of mass are related to one another as follows:

1 metric ton	=1000 kilograms
1 centigram	=0.01 gram
1 milligram	=0.001 gram
1 pound (avoirdupois)	=453.592 grams
1 short ton	=2000 lbs.
1 short ton	=907.185 kilograms
1 short ton	=0.907185 metric ton
1 long or gross ton	=2240 lbs.

13. C. G. S. or Absolute System of Units. — We have seen that the standard units adopted in scientific work for the measurement of the fundamental quantities of length, mass and time are the centimeter, gram and second. Units for the measurement of all other quantities such as surface, volume, velocity, acceleration, force, etc., can be expressed in terms of these units; such units are called *derived* units in contradistinction to the three *fundamental* or absolute units of length, mass and time. The system of units derived from the units of centimeter, gram and second is known as the *absolute system*, or the *c. g. s. system*, from the initials of the three fundamental units.

14. Density and Specific Gravity. — The *density* of a uniform substance is defined as the mass of the substance per unit volume.

In the *c. g. s.* system density is the weight in grams of one cubic centimeter of the substance. When the substance is not uniform, its density at any point is defined as the mass of an infinitesimally small volume taken about the point divided by this volume; *i.e.*, calling dv the volume and dm the mass of this volume, the density is

$$\delta = \frac{dm}{dv}. \tag{10}$$

The *specific gravity* of a substance is defined as the ratio of the weight of a given volume of the substance to an equal volume of water at standard temperature. Sixty-two degrees Fahrenheit is usually taken as the standard temperature, although there is no general agreement on this point. Density on the *c. g. s.* system and specific gravity are practically numerically equal.

15. Center of Mass.—A body of mass M which has any size or shape may be considered as made up of a number of small particles of masses m_1, m_2, m_3, etc., such that $m_1 + m_2 + m_3 + \text{——} = M$. These particles may be considered as small as we wish, that is we may consider each particle so small that it occupies but a point in space. If we consider three mutually perpendicular planes X, Y, Z, fixed in space, and represent by x_1, y_1, and z_1 the perpendicular distances of the particle m_1 from these planes respectively, and by x_2, y_2, and z_2 the perpendicular distances of the particle m_2 from these three planes respectively, and so on for the other particles, then the point whose distances from these three planes are respectively

$$\begin{aligned} x &= \frac{m_1x_1 + m_2x_2 + m_3x_3 + \text{——}}{M} \\ y &= \frac{m_1y_1 + m_2y_2 + m_3y_3 + \text{——}}{M} \\ z &= \frac{m_1z_1 + m_2z_2 + m_3z_3 + \text{——}}{M} \end{aligned} \tag{11}$$

is defined as the *center of mass* of the body. The center of mass of a body is therefore the point the distance of which from each of three mutually perpendicular planes is the average distance of the matter in the body from each of these planes. It can be shown that the position of the center of mass of a body relative to any

point in the body is independent of the position of the planes of reference.

The center of mass of a system of any number of bodies is defined in exactly the same manner, except that M in this case is taken as the *total* mass of *all* the bodies.

16. Linear Momentum and Moment of Momentum. — When a body as a whole is in motion, or when there is any relative motion of parts of the body, the various points of the body will in general move with different velocities. It is therefore convenient, in analysing the motion of a system of bodies, to consider each body as made up of a number of individual particles and to take these particles so small that the mass of each may be considered as occupying but a point in space. The product of the mass (m) of each particle times its linear velocity (v) is defined as the *linear momentum* of the particle, *i.e.*, linear momentum $=mv$. Linear momentum is a vector quantity, since v is a vector quantity and m is a scalar quantity. The vector sum of the linear momenta of all the particles of a rigid body is called the total linear momentum of the body. From the above definition of center of mass it can be shown that the total linear momentum of a body is equal to its total mass times the linear velocity of its center of mass.

Consider a fixed axis, and a particle of mass m at a distance r from the axis, and let the particle be moving with a velocity v. Then the product of m, r and that component u of the velocity v perpendicular to the plane passing through the particle and the axis, is defined as the *moment of momentum* of the particle about the fixed axis; *i.e.*, moment of momentum $=mru$. The moment of momentum of a particle is to be taken positive if the particle moves in a clockwise direction as seen by an observer looking along the axis in the positive sense, negative if in the opposite direction. The component u of the linear velocity of a particle at right angles to a plane passing through the particle and any fixed axis, is equal to the product of the distance r of the particle from the axis times the angular velocity ω of the particle about this axis; therefore the moment of momentum of the particle may also be written $mr^2\omega$. In the case of a rigid body each particle of which has the same angular velocity ω about a given axis, the moment of momentum about this axis is equal to $\omega\Sigma mr^2$, the summation including all the particles of the

body. The quantity Σmr^2 is called the *moment of inertia* of the body about the given axis of rotation, and is usually written I; then

$$I = \Sigma mr^2 \tag{12}$$

The moment of momentum of a solid body rotating in this manner is called the *angular momentum* of the body; angular momentum then equals $I\omega$. The moment of inertia of a body of given dimensions and given distribution of material about a given axis may be written Mk^2 where M is the total mass of the body and k is a length such that

$$k^2 = \frac{\Sigma mr^2}{M}. \tag{12a}$$

The length k is called the *radius of gyration* of the body about the given axis.

17. Conservation of Mass, Conservation of Linear Momentum, Conservation of Moment of Momentum. — It has been found that every phenomenon of nature, which has so far been tested by experiment, takes place in such a way that the three following conditions are invariably satisfied:

1. Matter cannot be created or destroyed. As a consequence of this condition, if any number of bodies are kept entirely separate from the rest of the universe, for example, in a closed vessel through the walls of which no matter can pass, then the total mass of these bodies must likewise remain constant, irrespective of any changes that may take place in these bodies. This condition is known as the **law or principle of the conservation of mass.**

2. When the linear momentum of one or more bodies changes relative to any fixed point, then there must be an equal and opposite change in the linear momentum of some other body or bodies relative to this point. This condition is known as the **law or principle of the conservation of linear momentum.**

3. When the moment of momentum of any body or bodies about any fixed axis changes, then there must be an equal and opposite change in the moment of momentum of some other body or bodies about this same axis. This condition is known as the **law or principle of the conservation of moment of momentum.**

18. Force. — In general terms, a force is that which produces or tends to produce a change in the motion of a body. Since, by the principle of the conservation of linear momentum, any change in the motion of a particle A is accompanied by a change in the motion of some other particle or particles B, it is convenient

to consider the force acting on A as due to the presence of the particle or particles B. Or, we may say that the change in the motion of A is due to a force produced on A by the particle or particles B. In other words, we may consider the changes in motion of material particles as being due to a property possessed by these particles themselves.* We take arbitrarily as the measure of the force with which a particle A is acted upon by a particle B the time rate of change of the linear momentum of A relative to any fixed point, when the relative motion of the two particles with respect to each other is unaffected by the presence of any other particle or particles; the direction of the force on A due to the particle B is defined as the direction of the rate of change of the linear momentum of A. From the principle of the conservation of linear momentum we then conclude that the force with which B is acted upon by A is equal and opposite to the force with which A is acted upon by B, that is, "action and reaction are equal and opposite." When the masses of the two particles A and B remain unchanged, the time rate of change of linear momentum of each particle is equal to the product of its mass and its acceleration, *i.e.*, the force with which A acts on B is ma and the force with which B acts on A is m_1a_1, where m and m_1 are the masses of A and B respectively and a and a_1 are the accelerations of A and B respectively. The direction of the force acting on A is then the direction of the linear acceleration of A, and the direction of the force acting on B is the direction of the linear acceleration of B. The acceleration of A will then

*This conception of the something that causes changes in the motion of matter as localized in material particles merely gives us a convenient way of describing experimentally observed facts. It is also conceivable that the changes in the motion of matter are due, not to a property inherent in matter itself, but to a property possessed by the medium or the "ether" in which the particles of matter are immersed. However, a particle of matter is a tangible thing, whereas the existence of the ether is purely hypothetical. For this reason we shall adhere to the older conception that every force, whether it be gravitational, electric, or magnetic, has its origin in a material particle and that it can make its influence felt on other particles at a distance from it. As far as engineering is concerned, we need not concern ourselves with the mechanism by which this action takes place, about which, at best, we can only theorize. Analogies, where they help one to form a mental picture of how certain effects *might* be produced, are extremely useful and will be frequently employed; but it must be remembered that analogies do not *explain* anything.

be in the opposite sense to the acceleration of B, and the ratio of the acceleration of A to that of B will be equal to the inverse ratio of the respective masses. Since acceleration is a vector quantity, force is likewise a vector quantity; therefore, when there are several forces acting on a particle, the resultant force on the particle is the vector sum of all the forces acting; this resultant force is also equal to the product of the mass of the particle by its acceleration in the direction of the force. The line through a particle in the direction of the force acting on it is called the line of action of the force.

The unit of force in the *c. g. s.* system of units is that force which will give unit linear acceleration (one centimeter per second per second) to a mass of one gram; this unit force is called the *dyne.* A similar unit in the foot-pound-second system, called the *poundal,* is sometimes used; it is defined as the force which will accelerate one pound one foot per second per second.

It is a matter of experience that each particle of matter which has neither electric nor magnetic properties (to be described later), when allowed to fall in a vacuum from any given height at any point near the earth's surface, falls to the earth in a vertical line with a constant acceleration independent of the size, shape, or material of the body. We therefore say that the earth exerts a force on each particle of the body proportional to its mass, and that therefore the total force with which the earth " attracts " a given body is equal to the total mass of the body multiplied by its acceleration when falling freely to the earth in a vacuum. We have seen (Article 11) that the acceleration " due to gravity " is constant for any given point with reference to the earth's surface, but varies slightly with the location and also with the elevation above the earth. For most practical purposes the acceleration due to gravity may be taken as 981. The attraction of the earth on a body offers a ready means for measuring a force, since it is only necessary to balance the force to be measured against the force exerted by the earth on a known mass, or by measuring the change produced by the force to be measured in the shape of some body, *e.g.*, a spiral spring, which has been previously "calibrated " by suspending known masses from it. These are the usual ways of measuring forces, since the determination of the acceleration due to any other force than " gravity " is extremely difficult.

When forces are measured in this way, they are usually expressed in terms of units which are given the same name as the corresponding units of mass. For example, by a force of one gram is meant the force with which the earth attracts one gram at sea level and 45° latitude; this is equal to a force of 980.5966 dynes, or approximately 981 dynes. Similarly, a force may be expressed as so many kilograms or so many pounds. It should be noted that forces specified in this way are not absolutely definite unless the place at which the force is measured is also stated; the variation at ordinary places of observation, however, is so slight that it is negligible in ordinary engineering work.

The relations between grams, kilograms, pounds, tons, etc., are the same whether these quantities are considered as masses or forces; these relations are given in Article 12. The units kilograms and pounds as forces are related to the dyne and the poundal at sea level and 45° latitude as follows:

1 kilogram	=980,596.6 dynes
1 kilogram	=70.927 poundals
1 pound	=444,791 dynes
1 pound	=32.172 poundals

19. Moment of Force or Torque. — Consider a fixed axis and a particle at a distance r from this axis. Let f_0 be the component of the force f acting on this particle perpendicular to the plane passing through the particle and the axis. The product f_0r is called the *moment of the force* f about this axis. It can be deduced from the principle of the conservation of moment of momentum that, when there are any number of external forces acting on a system of particles, the rate of change of the total moment of momentum of the system about any fixed axis is equal to the algebraic sum of the moments of all the forces about this axis, irrespective of any forces which the individual particles may exert on one another.

When a rigid body moves in such a manner that each point has at any instant the same angular velocity about a given axis, the rate of change of the angular momentum of the body about this axis is equal to the product of the moment of inertia (I) of the body about the given axis and the angular acceleration (α) about this axis; hence,

$$I\alpha = \Sigma f_0 r \qquad (13a)$$

The moment of a force about any axis is frequently called the *torque* about this axis; we then have that the algebraic sum of all the torques (T_1, T_2, T_3, etc.) about any axis is equal to the product of the moment of inertia (I) of the body about this axis and the angular acceleration (α) about this axis, that is

$$I\alpha = T_1 + T_2 + T_3 + \text{—}. \qquad (13b)$$

When the line of action of any force passes through the axis, the torque corresponding to this force is of course zero. When two forces are acting on a body, the condition for constant angular velocity is that the corresponding torque be equal and opposite.

The unit torque in the *c. g. s.* system of units is the torque produced on a particle by a force of one dyne acting perpendicular to the plane determined by the position of the particle and the axis of rotation and at a distance of one centimeter from the latter — this unit is called the *centimeter-dyne*. Other common units of torque are the *pound-foot* and the *centimeter-gram*. The relations of these units one to another are the same as for the units of energy (see Article 21) having corresponding names.

20. Motion of a System of Particles Acted upon by Several Forces.—When only one of a system of particles is acted upon by a force external to the system (for example, a stretched string attached to a point in a solid body) the particle on which the force is acting will in general exert a force on the other particles of the body and those particles in turn will each exert a force on the particle to which the force is applied. It can readily be shown that, as a consequence of the principle of the conservation of linear momentum, the resultant acceleration of the system of particles in this case will be such that the center of mass of the system will be given an acceleration equal to the external force acting on the particle divided by the total mass of the body; that is, a single external force acting on any particle of a system will produce the same acceleration of the center of mass of the system as would be produced by the same force acting on a single particle located at the center of mass of system and having a mass equal to the total mass of the system. In the case of several external forces acting on the system, the acceleration of the center of mass of the system will be equal to the resultant force divided by the total mass, or calling M the mass of the system, F the resultant of all the external forces acting on it, and A the linear acceleration of the center of mass, then

$$F = MA. \tag{14}$$

In general, each particle of the system will also have its angular velocity about any axis changed due to the action of a force on any one particle. In the case of a rigid body, it can be shown that in addition to the change in the velocity of the center of mass produced by this force, each particle of the body will be given an angular acceleration about an axis through the center of mass perpendicular to the plane determined by the center of mass and the line of application of the force, and that this angular acceleration will be equal to the moment of the force about this axis divided by the moment of inertia of the body about this axis. The condition for no angular acceleration is then that the line of action of the force (or of the resultant force when there is more than one force acting) pass through the center of mass; *vice versa,* when there is no angular acceleration of the body, the line of action of the force (or of the resultant force when there is more than one force acting) must pass through the center of mass.

When there are two equal and opposite external forces acting on a rigid body, there can be no linear acceleration of the body, *i.e.*, no acceleration of its center of mass. However, when the lines of action of these two forces do not coincide, the moment of the two forces about any axis will not be equal, hence there will be an angular acceleration of the body about its center of mass. The axis of this acceleration will be through the center of mass perpendicular to the plane determined by the line of action of the two forces, and the resultant moment of the two forces about this axis will be equal to the product of either force by the perpendicular distance between their lines of action. Two such equal and opposite forces are called a *couple*, and the value of the resultant moment is called the strength of the couple. Two couples balance each other, *i.e.*, there is no angular acceleration, when their strengths are equal.

21. Work and Energy. — Whenever one portion of matter effects a change in some other portion of matter, the former is said *to do work* on the latter. The attribute or condition of matter in virtue of which one portion of matter can effect changes in other portions of matter is called *energy;* that is, energy is the capacity for doing work. One means by which a body A may produce a change in another body B is by exerting a force on the latter and producing, as a result of this force, a displacement

of this body as a whole or a displacement of its individual parts; in this case the body A is said to do *mechanical* work on the body B. As the measure of the amount of mechanical work done on a particle of matter by the force which causes its displacement is taken the product of the displacement of the particle by the component of this force in the direction of the displacement, provided this component of the force remains constant during the displacement.

In general, the direction of the displacement as well as the amount and the direction of the force will change as the position of the particle is changed; in such a case, the above definition applies only to an infinitesimal displacement of the particle, *i.e.*, to a displacement so small that while the particle is being displaced this small amount, the force may be considered constant both in amount and direction. The mechanical work W done on a particle displaced any finite distance l is then the sum of the products of the force for each infinitesimal displacement times the component of this displacement in the direction of this force; *i.e.*,

$$W = \int_0^l (f \cos \theta)\, dl \qquad (15)$$

where f represents the force during any infinitesimal displacement, dl the displacement and θ the angle between the direction of the displacement and the direction of the force.

As the measure of the amount of work done on a body when a change is produced in it by other means than by a displacement of the body as a whole or by a displacement of its individual parts under the action of a force, is taken the amount of mechanical work which would be required to produce exactly this same change were this change effected solely by a displacement of the body as a whole or by a displacement of its individual parts by means of a force exerted on it. For example, when the temperature of a body is increased by any means whatever, the body is said to have an amount of work done on it equal to the amount of mechanical work which would have to be done on it to produce exactly this same rise of temperature.

The results of all known experiments justify the assumption that whenever work is done on a body this body is in turn given the capacity for doing an exactly equal amount of work on other bodies, and that the capacity of some other body or system of

bodies for doing work is diminished by an exactly equal amount. That is, whenever work is done, some body or system of bodies loses an amount of energy equal to the amount of work done and some body or system of bodies gains an exactly equal amount of energy. This assumption, which is justified by all known experiments, is known as the **principle of the conservation of energy.** This principle, together with the principles of the Conservation of Mass, the Conservation of Linear Momentum and the Conservation of Moment of Momentum, are the four cardinal principles of all natural science and engineering.

It is found convenient in discussing the various properties of matter to look upon each property of a portion of matter as representing a definite amount of energy and to give a special name to the energy associated with each property. For example, the energy associated with a body in virtue of its speed is called its *kinetic* energy; the energy associated with a body in virtue of its position with respect to other bodies which exert forces on it is called its *potential* energy; the energy possessed by a body in virtue of its temperature is called its *heat* energy or its *thermal* energy; the energy associated with a body in virtue of its chemical nature is called its *chemical* energy, etc. In this terminology the principle of the conservation of energy may be stated thus: The only possible changes which can take place in the energy associated with a system of bodies which neither influences, nor is influenced by, any other bodies, are changes in *form;* the *total amount* of energy in the system remains unaltered.

It can readily be shown that the kinetic energy of a particle of mass m moving with a linear speed s is $\frac{1}{2}\, m\, s^2$. A rigid body which is rotating with an angular speed ω, about an axis fixed in the body and passing through its center of mass, which axis at the same time has a linear speed s (for example, the armature of a railway motor moving relatively to the earth) has a total amount of kinetic energy equal to

$$\frac{1}{2}\, M\, s^2 + \frac{1}{2}\, I\, \omega^2 \tag{16}$$

relative to the earth, where M is the total mass of the body and I is the moment of inertia of the body about the axis of rotation fixed in the body. The expressions for other forms of energy will be given when the corresponding properties of matter are discussed.

The unit of work or energy on the *c. g. s.* system is the work done when a force of one dyne displaces a particle a distance of one centimeter; this unit is called the *erg*. Other common units of work and energy are related to one another and to the erg as follows:

1 gram-centimeter	=980.5966 ergs (at sea level and 45° lat.)
1 joule	=1 watt-second
1 joule	$=10^7$ ergs
1 kilogram-meter	$=10^5$ gram-centimeters
1 foot-pound	=1.35573 joules
1 foot-pound	=0.138255 kilogram-meter
1 small calorie	=0.001 large calorie
1 small calorie	=4.186 joules
1 small calorie	=3.088 foot-pounds
1 British thermal unit	=251.996 small calories
1 British thermal unit	=1,054.9 joules
1 British thermal unit	=778.1 foot-pounds
1 kilowatt-hour	=3,600,000 joules
1 kilowatt-hour	=2,655,400 foot-pounds
1 horsepower-hour	=2,684,300 joules
1 horsepower-hour	=1,980,000 foot-pounds

22. Power. — *Power* is defined as the time rate of doing work, or as the time rate of change of energy; the two definitions are equivalent.

When an amount of energy dW is transferred in an infinitesimal interval of time dt, then the corresponding power is

$$P=\frac{dW}{dt}. \tag{17}$$

When this rate of transfer of energy is constant, then the power may be defined as the amount of energy transferred, or the work done, in unit time. It should also be noted that since energy or work is the product of force and displacement, power may also be defined as the product of force and velocity.

In the *c. g. s.* system the unit of power is the power required to do work at the rate of one erg per second. This unit is seldom used as it is an extremely small quantity; instead is employed a unit which is 10,000,000 ergs per second, called the *watt;* one watt is therefore equal to one joule per second. Other common units

of power are related to one another and to the watt as follows:

1 kilowatt	=1000 watts
1 megawatt	=1000 kilowatts
1 metric horsepower	=75 kilogram-meters per second
1 metric horsepower	=0.735448 kilowatt
1 horsepower	=33,000 foot-pounds per minute
1 horsepower	=0.74565 kilowatt

It is to be noted that torque multiplied by angular velocity (in radians per unit time) gives the power of a rotating body expressed in the corresponding units. In stating the performance of electric machines use is frequently made of the expression " torque at one-foot radius "; this is equivalent to expressing the torque in pound-feet. (In the expression " torque at one-foot radius," the word torque is incorrectly used, since torque is independent of the radius — what is meant is the " force " at one foot radius.) The following are useful relations:

Power in kilowatts	$=1.4197\times10^{-4}\times$ torque in pound-feet $\times$ revolutions per minute.
Power in horsepower	$=1.9040\times10^{-4}\times$ torque in pound-feet $\times$ revolutions per minute.

23. Harmonic Motion. — A particular type of motion which is not only of considerable importance itself, but also serves as a useful analogue in the discussion of electric circuits, is that known as *harmonic* motion. Harmonic motion is defined as motion such that the acceleration of the moving body is proportional to, and in the *opposite* direction to, the displacement of the body from its position of equilibrium. (The equilibrium position of a body is that position in which the resultant force acting on the body is zero.) Harmonic motion is illustrated by the motion of a pendulum, the motion of a weight attached to a spring, the motion of each particle of a tuning fork, the motion of the balance wheel of a watch, etc. The first three are examples of harmonic motion of translation, the latter of harmonic motion of rotation.

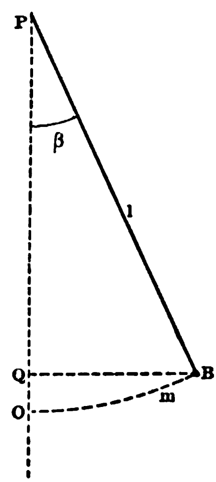


Fig. 8.

Consider the case of a simple pendulum. Let B be the bob of the pendulum and let it be so small that it may be considered

as a single particle of mass m, and let this bob be suspended by a weightless thread of length l from a fixed point P. Let the bob be displaced from its position of equilibrium and at any time t let its displacement, measured along the arc in which it moves, be $x = arc\ OB$. The linear velocity at which the bob is moving at any instant is then $\frac{dx}{dt}$ and its kinetic energy is therefore

$$\tfrac{1}{2} m \left(\frac{dx}{dt}\right)^2$$

The potential energy of the bob at this instant is

$$mg\ \overline{OQ} = mgl\,(1 - \cos\beta)$$

where $\beta = \frac{x}{l}$ and g is the acceleration due to gravity. The total energy of the bob is then

$$W = \tfrac{1}{2} m \left(\frac{dx}{dt}\right)^2 + mgl\,(1 - \cos\beta)$$

and if *there is no friction* this energy must be constant, and therefore the rate of change of this total energy with respect to time must be zero. Hence

$$\frac{dW}{dt} = m\,\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} + mgl\sin\beta.\ \frac{d\beta}{dt} = 0$$

whence, since $\frac{d\beta}{dt} = \frac{1}{l}\,\frac{dx}{dt}$,

$$\frac{d^2x}{dt^2} = -\,g\sin\beta$$

For β small, $\sin\beta = \beta$ when β is expressed in radians; whence for $\frac{x}{l}$ small, $\sin B = \frac{x}{l}$ and therefore

$$\frac{d^2x}{dt^2} = -\,\frac{g}{l}\,x. \tag{18}$$

Equation (18)* tells us that the linear acceleration of the bob along its path of motion is proportional to, and in the opposite direction to, its displacement; hence this is a case of harmonic motion.

*This equation may also be deduced directly from a consideration of the forces acting on the bob at any instant.

The solution of any differential equation of the form

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

where ω is a real constant, is

$$x = X_0 \sin(\omega t + \theta) \tag{19}$$

where X_0 is the maximum value of x, and θ is a constant angle determined by the value of x for $t=0$ and the maximum value X_0 of the variable x. Let x_0 be the value of x for $t=0$, then

$$\theta = \sin^{-1}\left(\frac{x_0}{X_0}\right) \tag{20}$$

This angle θ is called the *phase angle* of x, and measures the degree of "fullness" of x when $t=0$, just as the phase of the moon represents its degree of fullness. When $\theta=0$, x starts off with its zero value and increases; when $\theta=\frac{\pi}{2}$, x starts off with its maximum or full value and decreases. The constant ω depends upon the value of t required for x to pass through all possible values which it may take and back again to its original value; the amount by which t must increase in order that x may pass through all its possible values and back again is

$$T = \frac{2\pi}{\omega}. \tag{21}$$

This value T is called the *period* of x. The number of periods corresponding to an increase of unity in t (*e.g.*, if t represents time in seconds, the number of periods per second) is called the *frequency* of x; that is, the frequency is

$$f = \frac{\omega}{2\pi} \text{ or } \omega = 2\pi f. \tag{22}$$

Equation (19) may be represented by a sine curve, Fig. 9, where abscissas are ωt and ordinates x; the distance along the axis of ωt between the points where the curve cuts this axis in the *same* direction is equal to 2π radians or 360 degrees. The curve marked "x" in the figure is plotted for the case when $\theta = \frac{\pi}{2}$ that is, for x a maximum and decreasing at $\omega t=0$. The curve

marked " v " is for $\theta = \pi$. Note that the angle θ in any case is also equal to the distance to the *left* of the origin at which the curve first crosses the base line in the *positive* direction. A function of the form $X_0 \sin(\omega t + \theta)$, that is, a function which can be represented by a single sine wave, is sometimes called

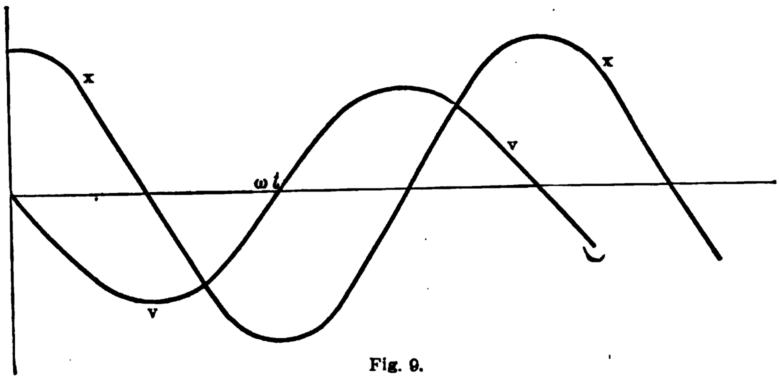


Fig. 9.

a *simple harmonic* function, or briefly, a *harmonic* function. Other periodic functions are called *non-harmonic* functions.

In the case of the simple pendulum let the bob be held out a distance A from its equilibrium position, and at a given instant let it be released. Let time be counted from this instant; then for $t = 0$ we have

$$x = A$$

and the velocity at this instant is

$$\frac{dx}{dt} = 0.$$

Substitution of these values in (19) gives

$$A = X_0 \sin \theta$$
$$0 = X_0 \cos \theta$$

whence $\theta = \frac{\pi}{2}$ and $X_0 = A$. Therefore the displacement of the bob from its equilibrium position at any instant is

$$x = A \sin\left(\omega t + \frac{\pi}{2}\right) = A \cos \omega t$$

and its velocity at this instant is

$$v = \frac{dx}{dt} = -\omega A \sin \omega t$$

where $\omega = \sqrt{\frac{g}{l}}$. Note that the displacement has its first positive maximum after the start for $\omega t = 2\pi$ and the velocity its first positive maximum for $\omega t = \frac{3\pi}{2}$; hence the velocity reaches a positive maximum a quarter of a period ahead of the displacement, or the velocity *leads* the displacement by 90° (see Fig. 9). Two sine waves of the same frequency which reach their positive maxima at different times are said to *differ in phase* by the angle corresponding to the time interval between successive positive maxima of the two waves. When both waves are expressed as sine functions, the difference in phase is the difference between the phase angles of the two functions. In the above example

$$x = A \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$v = \omega A \sin(\omega t + \pi)$$

whence the phase angle of x is $\frac{\pi}{2}$ and the phase angle of v is π; therefore the difference in phase between the two is

$$\pi - \frac{\pi}{2} = \frac{\pi}{2} \quad \text{or 90 degrees.}$$

The function with the larger (algebraically) phase angle always *leads*. Note that the *leading* curve is to the *left*.

24. Temperature. — The physical properties of any piece of matter depend, among other things, upon its temperature, *i.e.*, upon its relative hotness or coolness referred to some standard substance under standard conditions. The idea of temperature is familiar to every one, and one's so-called temperature sense enables one to form a rough judgment of the relative hotness or coolness of two or more bodies. For scientific purposes, however, a more reliable and more delicate means of "measuring" temperature is desirable. Any device which serves this purpose is called a thermometer.*

The standard temperature-measuring device is the constant volume hydrogen thermometer, which consists essentially of a suitable receptacle containing a constant mass of hydrogen gas kept at constant volume, with means provided for measuring any

*A thermometer designed to measure very high temperatures is called a "pyrometer."

variation that may be caused to take place in the pressure of the gas. The numerical value of the temperature of any substance is then defined in terms of the relative pressure of this gas when the receptacle is immersed in the substance, referred to the pressure of this gas when the receptacle is immersed in melting ice at a pressure of 760 mm. of mercury, the pressure of the gas in each case being measured after it has reached a constant value. The temperature of melting ice at a pressure of 760 mm. of mercury is arbitrarily taken as zero degrees, and the temperature of saturated steam at a pressure of 760 mm. of mercury is taken as 100 degrees. Calling p_1 the pressure of the hydrogen gas when the receptacle is immersed in the melting ice and p_2 its pressure when immersed in the saturated steam, and p its pressure when immersed in any given substance S (the pressure in each case being measured after the lapse of a sufficient time for it to reach a constant value), the numerical value of the temperature of the given substance is defined as $t = \frac{p - p_1}{p_2 - p_1} \times 100$ degrees centigrade.

The Fahrenheit scale of temperature is derived in the same manner, except that the temperature of the melting ice is taken as 32 degrees and that of the saturated steam as 212 degrees. A temperature of t_f degrees Fahrenheit is then equal to

$$t_c = \tfrac{5}{9}\,(t_f - 32) \text{ degrees centigrade.} \qquad (23)$$

For practical purposes, the volume expansion of mercury in glass, or of alcohol in glass, is used as a measure of temperature. For strictly accurate measurements, such mercury-in-glass or alcohol-in-glass thermometers should be calibrated by comparison with a standard hydrogen thermometer, but for ordinary work it is sufficient to determine the two points on the thermometer scale corresponding to the temperature of melting ice and saturated steam under standard conditions, and to assume that the volume expansion of the thermometer fluid is proportional to the temperature.

25. Heat Energy. — All known experiments lead us to believe that whenever the temperature of a body increases, energy is transferred *to* it, and that whenever the temperature of a body decreases energy is transferred *from* it; further, that there is a fixed numerical relation between the quantity of energy transferred to a given body and its change in temperature. This nu-

merical relation can be determined directly only in case the change in temperature is produced by mechanical work, since, by definition, the measure of energy is the product of a force by a distance. Careful experiments have shown that the work required to raise the temperature of one gram of water from 0° to 100° centigrade is 418.6×10^7 ergs. We may then take as the unit of heat energy the one-hundredth part of the work required to raise one gram of water from 0 to 100 degrees centigrade. This unit is known as the *mean small calorie,* and in terms of it, the amount of energy involved in various heat effects may be measured with comparative ease. It should be noted that the work required to raise one gram of water one degree is different at different temperatures; the variation is negligible, however, except in the most refined work.

The above numerical relation between the mean small calorie and the erg is called the *mechanical equivalent of heat* on the *c. g. s.* system. Another way of expressing the mechanical equivalent of heat is, the number of foot-pounds of work required to raise the temperature of one pound of water one degree Fahrenheit or at near its maximum density (39.1° Fahrenheit); this may likewise be used as a unit of heat energy, and is known as the "British Thermal Unit." See Article 21 for the relations between the various units of heat energy.

Practically every phenomenon in nature is accompanied by a change in temperature of one or more bodies. In certain cases the entire amount of energy transferred in the process may be caused to appear as heat energy, and when this can be done the total amount of energy transferred can be measured with a fair degree of accuracy. Usually, the heat energy developed in any process is not in a useful form; in such cases the heat energy is said to be "dissipated."

26. Efficiency and Losses. — In any machine or apparatus which is employed for transforming energy from one form into another or for transferring energy from one place to another, a certain amount of energy is always converted into forms which cannot be readily utilized. In general, this useless energy appears as heat energy. The rate at which energy is put into a machine is called the *power input* into the machine and the corresponding rate at which the machine gives out *useful* energy is called the *power output,* or the *load* on the machine. The difference between

the power input and the power output is called the *power loss.* The ratio of the output P_o to the input P_i for any given output of a machine is defined as the *efficiency* of the machine at this output; this ratio is usually expressed as a percentage. The per cent efficiency e is then

$$e=100\frac{P_o}{P_i}. \tag{24}$$

The ratio of the difference between the input P_i and the output P_o to the input P_i expressed as a percentage, is called the *per cent power loss p;* that is,

$$p=100\frac{P_i-P_o}{P_i}=100-e. \tag{25}$$

Sometimes it is more convenient to express the power loss as a percentage of the output P_o. Let p' be the percentage loss expressed in this manner; then

$$p'=100\frac{P_i-P_o}{P_o}=\frac{100p}{100-p} \tag{26}$$

The efficiency of a machine varies as a rule with the load or output. For no load, *i.e.*, no output, the efficiency is zero, since in general energy must be supplied to the machine to operate it, even though the machine does no useful work; as the load comes on the efficiency increases up to a certain output, depending on the design of the machine, and then decreases.

The *rated load* or rated power output of a machine which is designed for continuous service is the maximum rate at which useful energy may be transferred through the machine continuously without injury to any of its parts. In most electric machinery this is determined by the rise of temperature produced by the energy dissipated or "lost" in the windings and iron cores. As a rule, the insulation of the windings will deteriorate if the temperature of the machine exceeds 75° centigrade.

27. Newton's Law of Gravitation. — From the observations by various astronomers of the revolution of the planets about the sun, and of the revolution of the moon about the earth, Newton was led to the belief that every particle of matter in the universe attracts every other particle with a force proportional to the product of the masses of the two particles, and inversely proportional to the square of the distance between them, namely with a force

$$f=k\frac{mm'}{r^2} \tag{27}$$

where k is a constant depending upon the units in which m, m', r, and f are measured. Further observations and careful laboratory experiments by Cavendish and others have confirmed this belief which is now accepted as one of the fundamental "laws" of nature. The constant k has been determined by experiment to be 6.66×10^{-8} when m and m' are expressed in grams, r in centimeters, and f in dynes. For small masses, therefore, such as one ordinarily deals with in the laboratory, this force is extremely small, and can be detected only with the most delicate instruments.

As we shall see presently, *the mutual attraction or repulsion between magnetic poles and also the mutual attraction or repulsion between electric charges obey a law of exactly the same form.* Hence the study of forces which obey this law becomes of prime importance. It should be borne in mind that any conclusions derived from this law of force action will apply to any one of the three agents, gravitational masses, magnetic poles, and electric charges. In order to express such deductions in terms of physical quantities, we shall consider first the forces produced by magnetic poles.

PROBLEMS

1. Find the speed in meters per second and in feet per second of an electric locomotive travelling at a uniform speed of 60 miles per hour. If the locomotive is travelling at this speed around a curve of 1000 feet radius, what is the direction and amount of the acceleration in feet per second per second and in miles per hour per second?

Ans.: 26.8 meters per second; 88.0 feet per second; 7.74 feet per second per second; 5.28 miles per hour per second. Acceleration toward center of curvature.

2. In Problem 1, if the distance between rails is 4 feet, 8.5 inches, how many inches would the outer rail have to be elevated above the inner rail in order that the flanges of the locomotive wheels exert no side thrust on the rails?

Ans.: 13.60 inches. (In practice the elevation would be about half this; that is, the actual force on the outer rail would be about half what it would be were both rails in the same horizontal plane.)

3. If in Problem 1 the outer rail of the curve is elevated 6 inches and the locomotive weighs 100 tons, what will be the

average shearing force exerted on each of the spikes holding the rails to the ties? Assume the total force to be exerted against 15 spikes, and neglect the friction of the rail against the ties. Gauge of track 4 feet, 8.5 inches.

Ans.: 1794 pounds.

4. A man stands on the floor of a trolley car which is travelling at the rate of 20 miles an hour around a curve of 100 feet radius. How many degrees from the vertical must the man lean to prevent his falling over? Assume the man may be represented by a rod standing on its end. Draw a complete vector diagram of the forces acting.

Ans.: 15.0.°

5. The tangential force exerted by a brake on a pulley which is rotating at a speed of 500 revolutions per minute is 200 pounds. If the diameter of the pulley is one foot, calculate the torque in pound-feet developed by the pulley, and the amount of work in horsepower-hours done by the pulley in 10 minutes.

Ans.: 100 pound-feet; 1.587 horsepower-hours.

6. The radius of gyration of a cylinder about its own axis is equal to $\frac{D}{4}$ where D is the diameter of the cylinder. What is the moment of inertia of an iron cylinder 5 feet in diameter and 4 feet long? Specific gravity of iron 7.7; give answer in pound-foot units.

Ans.: 59,100 pound-foot^2.

7. If the cylinder in Problem 6 is rotating at a speed of 1000 revolutions per minute what is its kinetic energy in foot-pounds and in horsepower-hours?

Ans.: 324,000,000 foot-pounds; 163.6 horsepower-hours.

8. If the force driving the cylinder in Problem 7 is removed, how many hours will it take the cylinder to come to rest if the opposing torque due to friction is constant and equal to 300 pound-feet?

Ans.: 5.73 hours.

Note: The data given in Problems 6 to 8 are fairly representative of a large turbo-generator, except that the friction torque is not constant. Unless a brake of some kind is used the rotating part of such a machine will continue in motion for several hours after the steam is shut off.

9. A weight of 500 pounds falls from rest a distance of 100

feet into a tub of water. If the tub contains 10 gallons of water, and all the kinetic energy lost by the weight when it is brought to rest goes into heating the water, what will be the rise in the temperature of the water produced thereby, assuming no radiation of heat energy? Give answer in degrees centigrade and degrees Fahrenheit.

Ans.: 0.430° centigrade; 0.774° Fahrenheit.

10. The thermal efficiency of the best modern boiler is about 65 per cent; *i.e.*, of the total energy in the coal 65 per cent can be transferred to the steam. The efficiency of a modern steam engine working under the best conditions is about 20 per cent; *i.e.*, of the total energy in the steam which passes through the engine only 20 per cent is converted into mechanical energy. A pound of high-grade coal contains 14,000 British thermal units of energy. Assuming a boiler efficiency of 65 per cent and an engine efficiency of 20 per cent, how many pounds of coal will be required to produce one horsepower-hour at the engine shaft? What is the over all efficiency of boiler and engine?

Ans.: 1.40 pounds of coal per horsepower-hour; 13 per cent. (In practice a considerably greater amount of coal is required per horsepower-hour, due to the fact that an engine seldom works at full load all day and the efficiency is less for light loads. A large steam plant developing from 10,000 to 40,000 horsepower maximum load uses about 2 pounds of coal per horsepower hour; small plants of a few hundred horsepower capacity use about 5 pounds of coal per horsepower hour.)

11. If the efficiency of a water wheel is 80 per cent, how many cubic feet of water per second falling through a distance of one foot will be required to develop energy at the rate of one horsepower? Let Q be the number of cubic feet per second, H the "head" or distance through which the water falls, and P the horsepower; what is the relation between P, Q, and H for a wheel of 80 per cent efficiency?

Ans.: 11 cubic feet per second; $P = \frac{QH}{11}$.

12. A lead ball which has a mass of 1 pound is suspended by a light string from a fixed support; distance from center of mass of ball to support 1 yard. The ball is displaced a horizontal distance of 3 inches from the vertical and then let go. What is the expression for the displacement (in inches) of the ball from the

vertical t seconds later, neglecting the effect of friction; what will be its maximum velocity in inches per second and what will be its displacement from the vertical at this instant; what will be the frequency of the pendulum?

Ans.: $x = 3 \cos (3.27t)$ inches; 9.81 inches per second every time the displacement from the vertical is zero; 0.52 cycles per second.

13. Plot to scale the values of the displacement, velocity, and acceleration of the vibrating bob in Problem 12. What are the phase relations between the velocity and the displacement; between the velocity and the acceleration?

Ans.: The velocity leads the displacement by 90° and lags 90° behind the acceleration.

14. What will be the kinetic energy of the ball in Problem 12 when it is half way between its equilibrium position and its point of maximum deflection? What will be its potential energy at this instant? Give answers in joules and in foot-pounds.

Ans.: Kinetic energy 0.251 foot-pounds or 0.340 joules; potential energy 0.084 foot-pounds or 0.114 joules.

15. Two lead spheres each 1 foot in diameter are placed near each other with their nearest points 1 inch apart. The density of lead is 11.3. Calculate the force (gravitational) in dynes with which the two spheres attract each other, assuming the total mass of each sphere concentrated at its center.

Ans.: 1.73 dynes.

II

MAGNETISM

28. Magnets. — Magnetic Poles. — It has long been known that a certain mineral, called the loadstone, has the property of attracting pieces of iron or steel with a readily perceptible force, even though the loadstone and the piece of iron or steel be comparatively small. In other words, between loadstone and iron or steel there is a force of attraction many times greater than the force of attraction between like masses of ordinary matter. It has also been known for many centuries that a steel bar can be given this same property by stroking it lengthwise with a piece of loadstone, and that when the bar thus treated is freely suspended, it takes up a definite position with respect to the earth, one end of the bar pointing approximately toward the north geographical pole and the other end toward the south geographical pole. A steel bar having this property is called a *magnet*, the north pointing end is called its *north* or *positive pole* and the south pointing end its *south* or *negative pole*. In general, a magnet may be defined as a body which possesses the property of attracting with a readily perceptible force pieces of iron or steel, and which, when freely suspended, takes up a definite position with respect to the geographical meridian. As we shall see later on, it is possible to "magnetise" a steel bar in other ways than by stroking it with a piece of loadstone; in particular, when an insulated wire is wrapped around such a bar and an electric current is established in the wire, the bar becomes magnetised; this is the modern way of making a magnet and the only way of making a powerful one.

29. Paramagnetic and Diamagnetic Substances. — It is found by experiment that a magnet attracts not only iron and steel, but to a less extent nickel and cobalt. Bismuth, on the other hand, is repelled by a magnet. Substances which of themselves are not magnets but which are attracted by a magnet are called *paramagnetic substances*, or simply magnetic substances; substances which of themselves are not magnets but which are repelled by a magnet are called *diamagnetic substances*. Except in the case of iron, steel,

nickel, and cobalt, the force of attraction for paramagnetic bodies is extremely small; of the diamagnetic substances bismuth is the only one which is strongly repelled by a magnet, and the repulsion even of bismuth is weak compared to the attraction of iron.

Whether or not a substance is attracted or repelled by a magnet depends upon the nature of the medium surrounding the magnet and the substance. For example, it is found by experiment that a solution of perchloride of iron is attracted by a magnet when the solution and the magnet are surrounded by air. On the other hand, if the magnet is immersed in the solution and a bubble of air formed in the latter, the air will be repelled by the magnet. Therefore perchloride of iron is paramagnetic with respect to air, and air is diamagnetic with respect to perchloride of iron. In other words, paramagnetic and diamagnetic are purely relative terms. It is customary to take air as the standard of reference, although in certain cases it would be preferable to take free space or the "ether" as the standard. However, the difference between air and free space in respect to their magnetic qualities is practically inappreciable, so that a body which is magnetic with respect to air is likewise magnetic with respect to free space. In the following discussion, when the terms paramagnetic and diamagnetic are used, air is to be understood as the standard of reference; *i.e.*, air is *assumed* to be non-magnetic.

30. Attraction and Repulsion of Magnetic Poles. — A piece of iron or other magnetic substance which of itself is not a magnet, will be attracted by either pole of a magnet when placed near this pole. However, when two magnets are placed near each other, the mutual force between the two may be either an attraction or a repulsion. When the two like poles of the two magnets are nearer together than their unlike poles, the force is a repulsion; when the unlike poles are nearer together, the force is an attraction. This is readily tested by suspending one of the magnets by a thread attached to its middle point and bringing the two ends of the second magnet successively near one end of the suspended magnet. We therefore conclude that like magnetic poles repel each other and unlike poles attract each other. Experiment also shows that the force of attraction or repulsion of two magnets falls off rapidly as the distance between them is increased.

31. Magnetic Charge.— A Magnetic Pole as a Force-Producing Agent. — It is found by experiment that the forces produced by

magnets on one another and upon magnetic bodies placed in their vicinity may be accounted for in terms of a something confined solely to the external surfaces of such bodies. In the case of a long, slim magnet this force-producing agent is confined almost entirely to the ends of the magnet. Since the two ends of a magnet possess opposite properties, it is necessary to look upon this force-producing something as different at the two ends of the magnet. This something at the north pointing end of a magnet may be called a "positive magnetic charge," and the something at the south pointing end a "negative magnetic charge." However, it is customary to use the expression "north (or positive) magnetic *pole*" to signify this something at the north pointing end of a magnet and the expression "south (or negative) magnetic *pole*" to signify this something at the south pointing end. When used in this sense, the word "pole" signifies the force-producing agent associated with the surface of a magnet rather than the particular part of the magnet at which this force-producing agent is located.

It is to be noted that although the external forces produced by a magnet may be accounted for in terms of a something confined solely to its external surface, there is also a change produced throughout the substance of a body when it is magnetised. For, when such a body is broken in two, the two broken ends are found to be the seat of equal and opposite magnetic poles. There are also certain very special cases in which the poles of a magnetised body must be considered as distributed throughout its interior, but the theory developed below for surface poles may be readily extended to such cases.

32. Induced Magnetisation. — In order to express in an exact quantitative manner the value of the mutual forces produced on one another by magnetic poles it is always necessary to take into account the effect of any magnetic bodies which may be in their vicinity. As we have seen, when a magnetic body which itself is not a magnet (a small rod of soft iron, for example) is placed in the vicinity of a magnetic pole, this substance is attracted whether the pole is a north or a south pole. In fact, the body acts exactly like a magnet, except that the strength and position of its poles depend on its position with respect to the magnet attracting it. The magnetic body is therefore said to be magnetised by "induction." Since the magnetised body is always attracted, the loca-

tion of the poles induced on it is always such that two of the *unlike* poles on the attracting magnet and on the magnetised body respectively are nearer together than the like poles on these two bodies. A diamagnetic body placed in the vicinity of a magnet is similarly magnetised by induction, except that in this case, since there is a repulsion between the inducing magnet and the diamagnetic body, the *like* poles are nearer together. For example, a small rod AB of soft iron placed in the vicinity of a magnet in the various positions shown in Fig. 10, is magnetised as

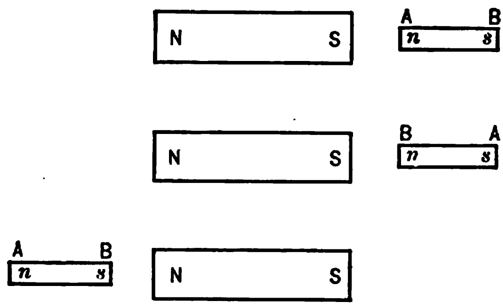


Fig. 10.

indicated. When the rod is removed from the vicinity of the magnet the induced poles disappear almost entirely. In case a hard steel rod, originally unmagnetised, is placed near the magnet, it becomes likewise magnetised by induction, though to a less extent than the soft iron. If, however, the steel rod is removed from the vicinity of the magnet it is found to retain to a considerable degree its magnetisation, that is, the rod becomes a "permanent" magnet.

The phenomenon of induced magnetisation is a most important one both in practice and in theory, and we shall return to this subject for a more detailed study. The chief fact to be borne in mind for the present is that *any magnetic or diamagnetic body placed in the vicinity of a magnet has magnetic poles induced on its surface, and that these induced poles produce forces on every magnetic pole, permanent or induced, which may be in the vicinity.* For example, when two magnets are separated from each other by a given distance and there are no magnetic or diamagnetic bodies in the vicinity, they will exert a certain force upon each other. When a magnetic body, such as a piece of unmagnetised soft iron, is placed

near them (see Fig. 11), the force acting on each magnet will evidently be changed, since the poles s and n induced on the

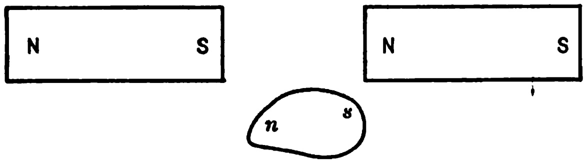


Fig. 11.

soft iron also exert forces upon each magnet. In the case illustrated, the resultant force acting on each magnet will be increased and also changed in direction. Again, when the entire region around the two magnets is filled with a magnetic liquid (perchloride of iron, for example), magnetic poles are induced on the surface of the liquid where it comes in contact with the poles of the magnets, Fig. 12, and these induced poles are of the opposite sign to those of the magnet at this surface. The effect of these induced poles in this case is to *decrease* the resultant force on each magnet, Art. 56. *The assumption that air is non-magnetic is equivalent to assuming that there are no poles, either permanent or induced, produced on the air in contact with a magnet.*

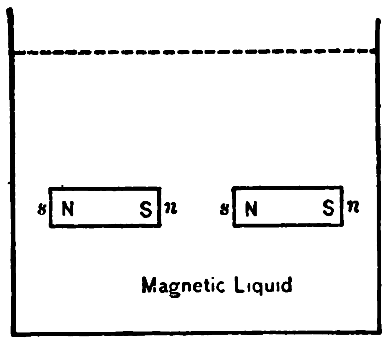


Fig. 12.

33. Point-Poles. — Experiment shows that it is physically impossible to have a magnetic " charge " or pole of finite amount concentrated in a point. However, for the purposes of mathematical analysis of the forces produced by magnets, it is frequently convenient to consider a magnetic pole as occupying but a point in space. We may call such a pole a *point-pole.* A physical approximation to such a point-pole is the pole on a very small area.

34. Properties of Magnetic Poles. — We are now in a position to state the properties which must be attributed to magnetic poles in order to account for the experimentally observed facts concerning their mutual action. When the *induced magnetic poles as well as the permanent poles are taken into account, and air is assumed to be non-magnetic* these properties are the following:

1. Like poles repel each other; unlike poles attract each other.

2. Whenever a magnetic pole of one sign exists on a body there is always an equal pole of the opposite sign on some other part of the *same* body. Neither kind of pole can be produced by itself.

3. Two point-poles of "strengths" m and m' respectively located a distance r apart *repel* each other with a force proportional to the product of the strengths m and m' and inversely proportional to the square of the distance between them; but *independent of the medium between them;* that is, with a force

$$f = k\frac{m\,m'}{r^2} \tag{1}$$

where k is a constant depending upon the units in which m, m', r, and f are measured.

Since we have not as yet specified what we shall take as a measure of the *strength* of a magnetic pole, we may select the unit of pole strength so that this constant k is unity when r is measured in centimeters and f in dynes. Equation (1) then becomes

$$f = \frac{m\,m'}{r^2} \tag{1a}$$

In this expression for the force between two magnetic poles the *strengths* of the poles m and m' are to be expressed as positive numbers when they are north or positive poles and as negative numbers when they are south or negative poles. Hence when m is a north pole and m' a south pole, or *vice versa*, the force of *repulsion* f is likewise negative, that is, the force f is an *attraction;* this is in accord with the first property attributed to these poles. The expression (1a) as thus understood may be looked upon as a definition of the measure of the strength of a magnetic pole. For, in accordance with this law of the mutual action of two point-poles, we may define the strengths of two such poles m and m' as equal when each repels with the same force f, a third point-pole m'' placed at the same distance from each. *A unit point-pole is then a point-pole which repels with a force of one dyne an equal point-pole placed one centimeter away.* This unit is called the *c. g. s.* electromagnetic unit of pole strength.

The above law of mutual action of two magnetic poles applies directly only to point-poles, for when the poles are extended over a surface the distance r between the poles has no definite meaning, since the distance between every pair of points taken respectively

in the two surfaces will be different. However, we may readily obtain an expression for the mutual action of two such poles in the following manner. Each pole may be considered as divided into a large number of areas so small that they may be considered as

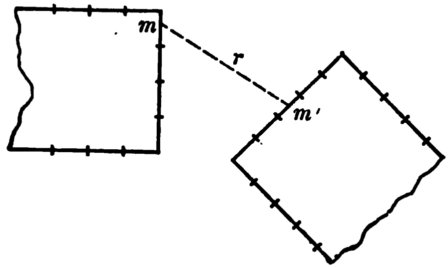


Fig. 13.

points, and we may call the pole strengths of these areas m_1, m_2, etc., for the first pole and m'_1, m'_2, etc., for the second pole, and r_{11}, r_{12}, etc., r_{21}, r_{22}, etc. the distances between m_1 and m'_1, m_1 and m'_2, etc., and m'_2 and m'_1, m_2 and m'_2, etc., respectively. The total force exerted by one pole on the other will then be

$$F=\frac{m_1 m'_1}{r^2_{11}}+\frac{m_1 m'_2}{r^2_{12}}+- - - - \frac{m_2 m'_1}{r^2_{21}}+\frac{m_2 m'_2}{r^2_{22}}+- - - \quad (1b)$$

where the terms of the right-hand side of the equation are added *vectorially*.

35. Pole Strength per Unit Area. — In order to calculate the actual value of this force it is of course necessary to know the value of the pole strength of each of these elementary surfaces; or, what amounts to the same thing, the pole strength per unit area at each point of the surface. Calling ds the area of any elementary surface and dm the pole strength of this surface, the *pole strength per unit area*, is then

$$\sigma=\frac{dm}{ds} \quad (2)$$

or the pole strength of this area is

$$dm=\sigma ds \quad (2a)$$

There are experimental methods for determining the value of the pole strength per unit area of a magnetised surface; one method is described in Article 37.

36. Magnetic Field of Force. — Field Intensity or Magnetising Force. — A magnetic pole placed anywhere in the region of space occupied by, or surrounding, a magnet, or in the vicinity of an

electric current (see Chapter III), will have a force exerted upon it; the region of space in which a magnetic pole is acted upon by a force is therefore called a *magnetic field of force.* In general, when a magnetic pole is placed in the vicinity of a magnet or of a magnetic body this pole will induce new poles on the magnet or magnetic body. Again, in order to place a magnetic pole inside a magnet it is necessary to cut a hole in the magnet, and, as pointed out in Article 46, whenever a hole or gap is cut in a magnet, magnetic poles in general appear on the walls of this gap. Hence, when a magnetic pole is placed near or inside a magnet or magnetic body, *new* poles will in general be formed. However, the force which would be exerted by the poles *originally in the field* on a pole placed at any point in the field, whether inside or outside the magnet, may be calculated from the fundamental law of the mutual action of two magnetic poles (equation 1*a*). Similarly, the force exerted by an electric current on a magnetic pole may be calculated from the fundamental law of the mutual action of a current and a pole (see Chapter III). *The force in dynes which would be exerted on a unit north point-pole placed at any point in a magnetic field, due solely to the poles and currents originally in the field, is defined as the intensity of the magnetic field at that point due to these agents.* The field intensity at any point is also called the *magnetising force* at that point.

It should be noted that the unit of field intensity is not the dyne, but is a *dyne per unit pole;* compare with power, the unit of which is not the erg, but is an *erg per second.* No specific name has been given to the *c. g. s.* unit of field intensity, but it is usually expressed as so many "gilberts per centimeter," the meaning of which expression is explained in Article 60. Magnetic field intensity or magnetising force is also expressed in terms of "ampere-turns per centimeter" or "ampere-turns per inch," the meaning of which expressions is explained in Chapter IV. The relations between these various units are

1 *c. g. s.* electromagnetic unit	=1 gilbert per centimeter
1 *c. g. s.* electromagnetic unit	=0.79578 ampere-turns per centimeter
1 *c. g. s.* electromagnetic unit	=2.0213 ampere-turns per inch

Since the force in dynes produced by a magnetic point-pole of strength m at a distance r centimeters away on a point-pole of strength

unity is $\frac{m \times 1}{r^2}$, the intensity of the magnetic field at a point P a distance of r centimeters from a point-pole of strength m, due solely to this pole, is

$$H = \frac{m}{r^2} \tag{3}$$

whether this point be outside or inside a magnet. The field intensity H will be in the direction of the line drawn from m to P where m is positive, and in the direction of the line drawn from P to m when m is negative. Field intensity is therefore a vector quantity.

The field intensity at any point due to any number of point-poles is then the vector sum

$$H = \overline{\Sigma} \frac{m}{r^2} \tag{3a}$$

where the symbol Σ *with a line over it* is used to represent a *vector* sum. Each pole then contributes a component $\frac{m}{r^2}$ to the resultant intensity, where m is the strength of this pole and r is its distance from the given point.

The force exerted on any point-pole of strength m in a magnetic field is

$$F = mH \tag{4}$$

where H is the field intensity at the point occupied by the pole m due to all the poles in the field *except* the pole m. The field due to any given pole can of course produce no mechanical force on this pole itself; " a man cannot lift himself by his boot-straps."

When the field intensity has the same value at every point throughout a given region, the field is said to be *uniform* throughout that region; we shall find several examples of practically uniform magnetic fields.

37. Example Illustrating the Application of Above Definitions.— The following example will serve to illustrate the ideas just discussed. The problem chosen is to find the field intensity at any point on the axis of a magnet having the shape of a long right cylinder and having its poles confined entirely to the end surfaces of the cylinder and of the same strength per unit area at each point of these end surfaces. Let σ be the strength per unit area of the north pole of this magnet and $-\sigma$ the pole strength per unit area of its south pole. Consider first the field

intensity due to its north pole. Let the point P, Fig. 14, at which the intensity is to be determined, be at a distance a from this pole, and let the radius of the magnet be r. Consider a ring drawn in the end surface with its center at N on the axis of the magnet, and let the radius of this ring be x and its width be dx. The elementary pole at any small area ds of this ring will produce a field intensity at P, and this force will have a component in the direction NP and a component perpendicular to NP. The elementary pole on an equal small area ds' diametrically opposite ds will likewise produce a field intensity at P numerically equal to that produced by the

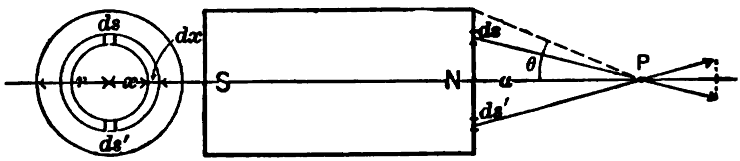


Fig. 14.

pole at ds, but in the direction shown. Hence the components of the field intensities, due to the poles at ds and ds', perpendicular to NP are exactly equal and opposite, and therefore exactly neutralise each other; similarly for any other two diametrically opposite points in this ring. Hence the resultant intensity due to the entire ring is in the direction NP and is equal to

$$\frac{\sigma \times 2\,\pi\, x dx}{a^2+x^2} \cdot \frac{a}{\sqrt{a^2+x^2}}$$

For $2\,\pi\, xdx$ is the area of the ring, and therefore $\sigma \times 2\,\pi\, xdx$ is the total pole strength of the ring; a^2+x^2 is the square of the distance of the point P from each element of the ring, and $\frac{a}{\sqrt{a^2+x^2}}$ is the cosine of the angle between the field intensity at P due to each point in the ring and the line NP. The total field intensity H_n, at P, due to the entire *north* pole of the magnet is then the sum of the field intensities due to all the contiguous elementary rings into which the pole face N may be divided. Hence

$$H_n = 2\,\pi\,\sigma\, a \int_{x=0}^{x=r} \frac{x\,dx}{(a^2+x^2)^{\frac{3}{2}}} = \pi\,\sigma\, a \int_{x=0}^{x=r} (a^2+x^2)^{-\frac{3}{2}}\, d(a^2+x^2)$$

$$= \pi\,\sigma\,a\left[-2(a^2+x^2)^{-\frac{1}{2}}\right]_0^r = 2\,\pi\,\sigma\left[1-\frac{a}{\sqrt{a^2+r^2}}\right]$$
$$= 2\,\pi\,\sigma(1-\cos\theta) \tag{5}$$

where θ is the angle between the axis of the magnet and the line drawn from P to the edge of its north pole. The field intensity due to the south pole of the magnet is similarly

$$H_s = -2\,\pi\,\sigma(1-\cos\theta')$$

where θ' is the angle between the axis of the magnet and the line drawn from P to the edge of its south pole.

Consequently the total field intensity is

$$H = H_n + H_s = 2\,\pi\,\sigma\,(\cos\theta' - \cos\theta). \tag{5a}$$

Each end of this magnet is a circular disc over which is uniformly distributed a magnetic pole. It is interesting to note what are the limiting values of the field intensity due to a single magnetically charged disc* when the point P is (1) very close to the disc, and (2) when the point P is at a considerable distance from the disc. In the first case, $\cos\theta$ becomes negligibly small in comparison with unity, since θ becomes practically 90°. Hence at the point on the axis of a uniformly magnetically charged disc at an infinitesimal distance from its surface, the field intensity *due to this disc alone* is

$$H = 2\,\pi\,\sigma \tag{5b}$$

and is independent of the size of the disc, but depends only upon its pole strength per unit area.

It can also be shown that a magnetic pole distributed in any manner over any *plane* surface, produces a field intensity at a point just outside this surface which has a component *normal* to this surface at this point equal to

$$H_n = 2\,\pi\,\sigma \tag{5c}$$

independent of the shape or size of the surface, where σ is the pole strength per unit area of the surface directly opposite this point. In general, however, there will also be a tangential component to the field intensity, that is, a component parallel to the surface; also the other pole of the magnet will be sufficiently close to produce a field intensity at this point, which intensity may have both a normal and a tangential component.

**e. g.*, the end of a very long cylindrical magnet when the other end is so remote that it produces no appreciable effect.

When the point P is at a great distance from the pole face, so that a is large compared with r, the angle θ becomes very small and therefore this angle in radians is equal to its tangent, *i.e.*,

$$\theta = \frac{r}{a}$$

Whence, expanding $\cos\theta$ into a series and neglecting all terms of a higher order than the second, we have

$$\cos\theta = 1 - \tfrac{1}{2}\left(\frac{r}{a}\right)^2$$

whence

$$1 - \cos\theta = \frac{r^2}{2a^2}$$

and therefore equation (5) becomes

$$H = \frac{\pi r^2 \sigma}{a^2} = \frac{m}{a^2} \qquad (5d)$$

where $m = \pi r^2 \sigma$ is the total pole strength.

Hence the field intensity at a point on the axis of a magnet a considerable distance from the magnet may be calculated approximately by assuming the poles concentrated in points at the ends of the magnet. In most practical problems a sufficiently accurate value of the field intensity at *any* point due to a bar magnet may be obtained by considering the poles of the magnet to be concentrated in points at its ends, *except* when the point under consideration is very close to a pole, but the limitations of this assumption should always be borne in mind.

38. Force Required to Separate Two Equal and Opposite Poles in Contact. — A useful application of equation (5*b*) is the calculation of the force required to separate two equal and opposite poles distributed uniformly over plane surfaces, such as the two poles of two equal bar magnets placed end to end. Call the two surfaces A and B and let them be separated by an infinitesimally narrow gap. Then, if σ is the pole strength per unit area of the north pole on the wall of the gap, the field intensity inside the gap produced by either surface is $H = 2\pi\sigma$, and therefore this surface attracts each unit pole on the other surface B with a force of $2\pi\sigma$ dynes. Hence, calling S the area of each surface, the total pole strength of the surface B is $-\sigma S$ and consequently the surface A attracts the surface B with a force of

$$F = 2\pi\sigma^2 S \qquad (6)$$

dynes. This formula is deduced on the assumption that the only forces acting are those due the two poles which are separated; if there are any other forces present (for example, the forces due to the other ends of the magnets and to the direct action of the electric current in the coil surrounding the magnets if the latter are electromagnets) their effect also must be taken into account. The formula is also based upon the assumption that the poles are confined solely to the end surfaces and are uniformly distributed over these end surfaces, which is practically never realized. The formula does give a rough means, however, for determining the average value of the pole strength per unit area, since both the force F and the area S are readily measured.

39. The Earth's Magnetic Field. — Since a magnet suspended at any point near the earth's surface tends to point in a definite direction, indicating the existence of a force acting on the magnet, the region of space in the vicinity of the earth is a magnetic field of force. For points at a considerable distance from any large masses of iron or wires carrying electric currents, the intensity of the earth's field is practically uniform over a large area, though there is a considerable variation in both the magnitude and the direction of the earth's field with longitude and latitude, and also a small variation from day to day and from year to year. The earth's field at any point in general has both a horizontal and a vertical component (downwards in the northern hemisphere, approximately). The horizontal intensity of the earth's field in the longitude of Washington and at latitude 45° north is roughly 0.2 *c. g. s.* electromagnetic units (defined above) and the vertical component 0.55 *c. g. s.* electromagnetic units. In any laboratory, however, particularly in the vicinity of electric trolley lines, the value of the magnetic field may differ considerably from this and varies continually.

40. Magnetic Moment. — Equivalent Length of a Magnet. — To hold a magnet in a uniform magnetic field in any other direction than that which it would take if acted upon only by the forces due to this field, requires a certain torque or moment, and the maximum value of this moment will correspond to a position of the magnet at right angles to its equilibrium position when acted upon only by the forces due to the field. The ratio of the value of the maximum moment which can be exerted by a *uniform* magnetic field on a given magnet to the value of the field intensity, is

called the *magnetic moment* of the magnet. In the ideal case of a magnet which has its poles concentrated in two points a distance l apart, the magnetic moment is equal to ml where m is the strength of the north pole of the magnet. For, the total force acting on each pole of such a magnet when placed in a uniform field of intensity H is mH, and when the magnet is at right angles to this field the moment acting on it is mHl, whence the ratio of this maximum moment to the field intensity H is ml. The *equivalent length* of any magnet is defined as the ratio of its magnetic moment to the total strength of its north pole.

41. Equality of the Poles of a Magnet. — Equilibrium Position of a Magnet in a Uniform Magnetic Field. — When a magnet of any shape is supported in a uniform magnetic field, for example, that due to the earth, in such a manner that it is free to move (*e.g.*, floated on a piece of cork), the magnet takes up a definite position and remains at rest. Hence the two poles of the magnet must be of equal and opposite strength; for, calling H the intensity of the field and m and m' the total strengths of the two poles respectively, the force acting on one pole of the magnet is mH and the force acting on the other pole is $m'H$, and therefore the total force on the magnet is $(m+m')\ H$. But since there is no motion of the magnet relative to the earth, which produces the field, this total force must be zero; hence $m=-m'$. Further, since there is no rotation of the magnet, its position of equilibrium must be such that the line of action of the two forces mH and $m'H$ coincides. Hence, when the magnet is in the shape of a long rod or needle, a line drawn from its south to its north pole will give the direction of the field. This fact is usually expressed by saying that the needle points in the direction of the magnetic field. When the motion of the magnet is limited to a definite plane, for example, when it is floated on a piece of cork or mounted on a pivot of the kind used in a magnetic compass, it will point in a direction corresponding to the *component* of the field intensity in this plane.

42. Measurement of the Horizontal Component of the Intensity of a Magnetic Field. — When a bar magnet or magnetic needle is supported in a magnetic field in such a manner that it is free to vibrate about a vertical axis, and is set vibrating about this axis, it will oscillate with a definite period depending upon the horizontal component of the intensity of the magnetic field; due to the friction of the air and of its support it will ultimately come to rest in line with the horizontal component of the field intensity. Let this hori-

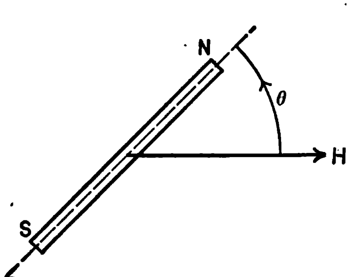


Fig. 15.

zontal component be H (Fig. 15), and let the magnet be so small that the intensity H is constant over the space occupied by it. Let the magnetic moment of the magnet be M, and the angular displacement of the magnet from its position of equilibrium (the direction of H) at any instant be θ. Let m be the numerical value of the strength of each pole of the magnet, and l its equivalent length. Then the torque acting on the magnet, neglecting the friction of the air and the support, tending to bring it back to its position of equilibrium is $ml\,H\sin\theta$. When θ is small, $\sin\theta$ is approximately equal to θ, and the torque is then approximately $mlH\theta$ or $MH\theta$. This torque is in such a direction as to oppose an *increase* in the angular displacement of the magnet, consequently the angular acceleration of the magnet about the vertical axis, neglecting the damping effect due to friction, is

$$\frac{d\theta^2}{dt^2} = -\frac{M\,H}{I}\theta$$

where I is the moment of inertia of the magnet (see Article 19). This equation shows that the motion of the magnet is harmonic (see Article 23) *i.e.*, the magnet will oscillate about its equilibrium position with a period equal to

$$2\pi\sqrt{\frac{I}{M\;H}}$$

The frequency with which the magnet vibrates is then

$$f=\frac{1}{2\pi}\sqrt{\frac{M\,H}{I}}$$

whence the intensity of the horizontal component of the field is

$$H=\frac{4\pi^2 If^2}{M} \qquad (7)$$

Note that the intensity H is proportional to the square of the frequency of vibration of the magnet. Hence, if the frequencies of vibration of the same magnet when suspended successively in two magnetic fields are f_1 and f_2, the ratio of the horizontal intensities of these two fields is

$$\frac{H_1}{H^2}=\frac{f^2_1}{f^2_2}$$

which gives a simple method of comparing the horizontal intensities of two magnetic fields.

In order to calculate H from the formula (7), it is necessary to know in addition to the frequency of vibration f, the moment of

inertia I and the magnetic moment M of the magnet. The moment of inertia I can be readily calculated when the magnet is in the form of a bar of rectangular cross section or in the form of a right cylinder, or it may be determined experimentally. A second relation between the moment M and the field intensity H can be obtained from the following experiment. Remove the bar magnet from the point P and suspend in its place a small magnetic needle (Fig. 16). Place the bar magnet with its center at a distance a from P with its axis perpendicular to the direction of H and in the same horizontal plane as the point P. The magnetic needle will now be acted upon by two magnetic fields, the original field H and the field due to the magnet. The intensity of this latter field will be

$$h=\frac{m}{\left(a-\frac{l}{2}\right)^2}-\frac{m}{\left(a+\frac{l}{2}\right)^2}$$

$$=\frac{2\,ml}{a^3\left[1-\left(\frac{l}{2a}\right)^2\right]^2}$$

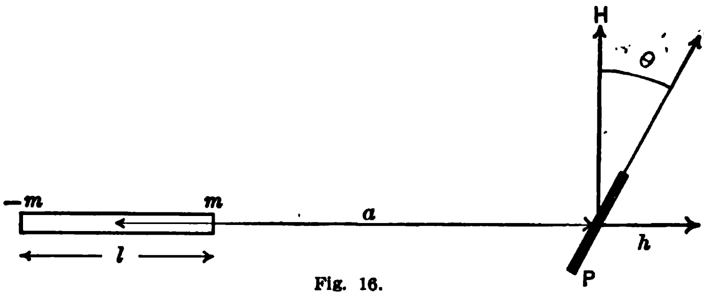


Fig. 16.

When a is chosen large in comparison with $\frac{l}{2}$ then $\left(\frac{l}{2a}\right)^2$ will be practically negligible in comparison with unity; hence to a close approximation

$$h=\frac{2ml}{a^3}=\frac{2M}{a^3}$$

where M is the moment of the bar magnet. When the magnetic needle is taken sufficiently small the value of the field intensity at the needle due to the bar magnet will be practically uniform over the space occupied by the needle, and therefore the resultant field intensity at the needle will also be uniform and will make an angle $\theta=tan^{-1}\frac{h}{H}$ with the direction of the original field H. This angle may be readily measured by noting the deflection of the needle when the bar magnet is removed to a great distance, for when

the bar magnet is present, the needle points in the direction of this resultant field intensity, $\overline{h+H}$, and when the bar magnet is removed the needle points in the direction of the original intensity H. Hence, the value of h in terms of θ and H is

$$h = H \tan \theta$$

whence

$$\frac{2M}{a^3} = H \tan \theta$$

or

$$M = \tfrac{1}{2} a^3 H \tan \theta$$

which substituted in the above equation for H gives

$$H = \frac{2\pi f}{a} \sqrt{\frac{2I}{a \tan \theta}} \qquad (7a)$$

and all the quantities in the right-hand member can be measured or calculated. Similarly, the substitution of this value for H in the equation just deduced for M gives

$$M = \pi a f \sqrt{2a I \tan \theta} \qquad (7b)$$

from which the magnetic moment may be calculated.

When the horizontal component of the field intensity has thus been determined, the direction and numerical value of the resultant intensity can be found by placing at the point P a magnetic needle which can turn only about a horizontal axis (a so-called "dip circle"), and setting this horizontal axis perpendicular to the horizontal component to the field. This needle will then point in the direction of the resultant field intensity, and if the angle it makes with the horizontal be α, the value of the resultant intensity is

$$H_r = \frac{H}{\cos \alpha} \qquad (7c)$$

It should be noted that there are other and more convenient methods for the measurement of the intensity of a magnetic field, based on the measurement of an electric current. But since as the measure of an electric current is taken the force produced by a magnetic field on a wire in which the current is established, it is necessary to start with an independent method for the measurement of magnetic field intensity.

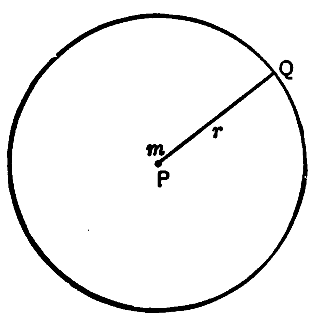


Fig. 17

43. Flux of Magnetic Force Due to a Single Pole.— Lines of Magnetic Force Due to a Single Pole. — Consider a point-pole m at any point P (Fig. 17). The field intensity at any point Q at a distance r from m is $\frac{m}{r^2}$ and is in the direction PQ

when the pole is positive or in the direction QP when the pole is negative. Consequently at every point in the surface of a sphere of radius r drawn about m as a center the field intensity will have the same value $\frac{m}{r^2}$ and will be normal to the surface of the sphere. Hence the product of the area of this sphere and the field intensity at its surface is

$$4\,\pi\,r^2.\frac{m}{r^2}=4\,\pi\,m$$

which is independent of the radius of the sphere but depends only upon the strength of the pole m. The product of the area of any sphere drawn about a point-pole as its center and the field intensity at the surface of this sphere is called the *flux of force* due to this pole, and this flux of force is said to be outward from the pole when the pole is positive and inward towards the pole when the pole is negative. A magnetic pole of strength m then produces a flux of force

$$\psi=4\,\pi m \qquad \text{(S)}$$

outward. (When m is negative the flux outward is also negative, which is equivalent to a *positive* flux *inward*.)

The flux of force from a magnetic point-pole can be represented graphically in a simple manner. Imagine the surface of any sphere surrounding the pole divided into $4\,\pi\,m$ equal areas and cones drawn with these areas as their bases and their common vertex at the pole m, and let the lateral walls of these cones extend out indefinitely. The number of these cones then represents the flux of force from the pole m. Each of these cones may be represented by a line coinciding with its axis, and the number of these lines will also represent the flux of force from the pole m, and these lines will coincide in direction with the field intensity at every point in their path. Such lines are called *magnetic lines of force.*

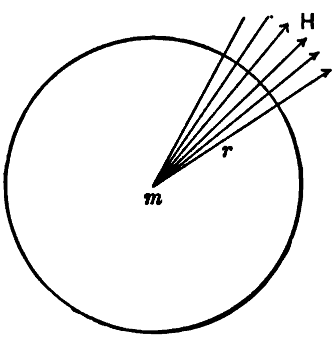


Fig. 18.

From the definition of these lines of force, it follows that the number of these lines per square centimeter of area normal to their direction at any point will be equal to the field intensity at that point;

for the unit area normal to their direction at any point Q at a distance r from the pole will be unit area in the surface of a sphere of radius r, and since a total of $4\ \pi m$ equally spaced lines are considered as radiating out from the pole m, the number crossing this unit area is equal to the total number of these lines divided by the total surface of the sphere, that is $\frac{4\pi m}{4\pi r^2}=\frac{m}{r^2}$. Hence the lines of force due to a point-pole m coincide in direction at every point with the direction of the field intensity at that point due to this pole and the number of these lines crossing unit area *at right angles to their direction* is equal to the field intensity at this area due to the pole m. The lines due to a single north pole radiate out to infinity; those due to a south pole radiate in from infinity.

It should be borne in mind that these lines really represent cones, and therefore it is entirely logical to speak of a fraction of a line of force. For example, from a pole of strength $\frac{1}{4\pi}$ there would be but a single cone, which would fill all space; that is, the cone would be a sphere with m at its center. A unit area normal to the direction of the field intensity at a distance of 10 centimeters from this pole would cut out $\frac{1}{4\pi\times10^2}=\frac{1}{1256}$th of this cone, or this unit area would be cut by only $\frac{1}{1256}$th of this cone of force, which, from the manner in which these cones are drawn, is equal to the field intensity at this point.

A simple relation also exists between the number of these lines of force crossing *any* area and the field intensity *normal* to that area. Let ds (Fig. 19) be any plane area at any point P taken

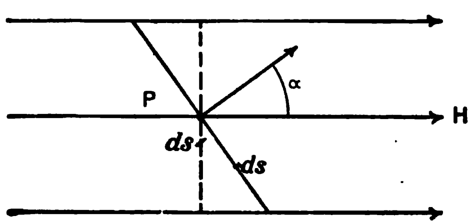


Fig. 19.

so small that the field intensity at every point in this area may be considered as having the same value and the same direction, and

let the normal to this area make the angle a with the direction of the field intensity, that is, with the direction of the lines of force through ds. Let ds' be the projection of ds on a plane through P perpendicular to the direction of the field intensity. Then the number, $d\psi$, of lines of force crossing ds' must be the same as the number crossing ds, since the perpendiculars dropped from the periphery of ds on the plane of ds' are parallel to the lines of force. But the field intensity H at the point P is equal to the number of lines of force per unit area crossing ds', that is

$$H=\frac{d\psi}{ds'}.$$

But $ds'=ds.\ cos\ a$, hence

$$H=\frac{d\psi}{ds.\ cos\ a}$$

or
$$H\ cos\ a=\frac{d\psi}{ds}$$

Therefore the number of lines of force per unit area at any point in a magnetic field, due to a point-pole of strength m, is equal to the *component* of the field intensity at this point *normal* to this area.

An important point to be noted in regard to a line of force is that a line crossing a surface in the direction from the side A to the side B is to be considered as equivalent to a *negative* line crossing

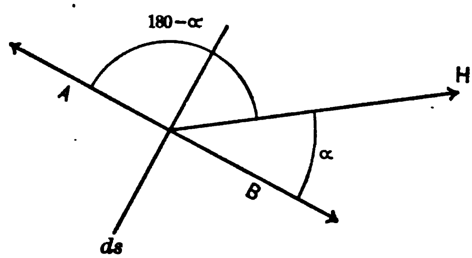


Fig. 20.

this surface in the direction from the side B to the side A. For, calling a the angle between the direction of the field intensity H and the normal drawn outward from B, then the angle between the direction of the field intensity and the normal drawn outward from A is $180°-a$. Hence the number of lines of force crossing the surface ds is $(H\ cos\ a)\ ds$ in the direction from A to B, or is $[H\ cos\ (180°-a)]\ ds=-(H\ cos\ a)\ ds$ in the direction from B to A.

44. Gauss's Theorem.—An extremely useful relation in the

theory of magnetism is that *the algebraic sum of the lines of force outward across any closed surface of any shape whatever is equal to* 4π *times the algebraic sum of the strengths of all the poles inside this surface.* This follows immediately from the fact that from any north pole of strength m there radiate *out* $4\pi m$ lines of force and therefore all these $4\pi m$ lines of force from a north pole inside such a surface will cut this surface in the direction from the inside outward, while to a south pole of strength m' there radiate *in* $4\pi m'$ lines of force, and therefore all these $4\pi m'$ lines of force radiating into a south pole inside such a surface cut this surface in the direction from the outside inward. Also, the lines of force due to any pole outside this surface, whether the pole be a north or a south pole, will either not cut the surface at all or will cut it an even number of times, as many times in the direction from outside inward as in the direction from inside outward. Since a line of force inward is equivalent to a negative line of force outward, in calculating the algebraic sum of the lines of force outward the lines of force due to the south poles inside the surface are to be substracted from the lines of force due to the north poles inside the surface; the lines of force due to any pole outside the surface contribute nothing to the algebraic sum of the lines of force outward, since each line leaves the surface as many times as it enters it. Hence the algebraic sum of the lines of force which cut a closed surface in the direction from the inside outward is equal to 4π times the *algebraic* sum of the strengths of all the poles inside this surface.

45. Lines of Force Representing the Resultant Field Due to any Number of Magnets. — So far we have been considering the lines of force due to each point-pole as a separate set of lines, there being as many sets of these lines as there are individual point-poles. In any actual case, however, whenever there is one magnetic pole there must be somewhere else in the field an opposite pole of equal strength. That is, in any actual case, the sum of the strengths of all the north poles in the field is equal to the sum of the strengths of all the south poles in the field. A field due to

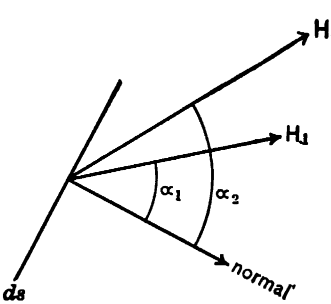


Fig. 21.

any number of such equal and opposite poles may also be represented by a *single set* of lines, drawn out from the north poles in the field, 4π of these "resultant" lines being drawn from each unit north pole, and each line coinciding in direction at each point with the resultant field intensity at that point. All of these lines will also end on south poles, 4π of them on every unit south pole. Also, the number of these resultant lines of force per unit area crossing any elementary area ds at any point in the field is equal to the component H_n of the resultant field intensity normal to this area, *i.e.*

$$H_n = \frac{d\psi}{ds} \tag{9}$$

when the area ds is taken normal to the direction of the field intensity the number of lines of force per unit area is equal to the *resultant* intensity H, since in this case the normal component is equal to the resultant.

Consider first the number of lines of force, considered as separate sets of radial lines from each pole, which cross any elementary area ds in a magnetic field due to any number of poles. In Fig. 21 let H_1 H_2 etc., be the field intensities at ds due to the individual poles producing the field, and let α_1, α_2, etc., be the angles between the normal to ds and the directions of the respective field intensities at ds. Then the number of lines of force crossing ds in the direction of this normal to ds due to the first pole is $(H_1 \cos \alpha_1)$ ds; the number of lines of force crossing ds in this same direction due to the second pole is $(H_2 \cos \alpha_2)$ ds; etc. Therefore the total number of these lines of force crossing ds in direction of the normal is

$$d\psi = (H_1 \cos \alpha_1 + H_2 \cos \alpha_2 + - - -)\, ds$$

But $$H_1 \cos \alpha_1 + H_2 \cos \alpha_2 + - - - - - = H \cos \alpha$$

where H is the *resultant* field intensity at ds and α is the angle between the normal to ds and the direction of this *resultant* field intensity. Hence the net number of lines of force crossing ds in this direction may also be written

$$d\psi = (H \cos \alpha)\, ds \tag{10}$$

and therefore the net number of lines of force crossing any surface S is

$$\psi = \int_S (H \cos \alpha)\, ds \tag{10a}$$

where H is the *resultant* field intensity at any element ds of this

surface and a is the angle between the outward normal to this surface at ds and the direction of the resultant field intensity at ds, and $\int_S$ indicates the sum or integral of $(H \cos a)ds$ for all the elementary areas ds into which the surface is divided.

To prove that the resultant field intensity H may be represented by a single set of lines, let a closed surface A (Fig. 22) be drawn in the field in such a manner that it encloses all the *north* poles but none of the south poles. Then, by Gauss's Theorem, the

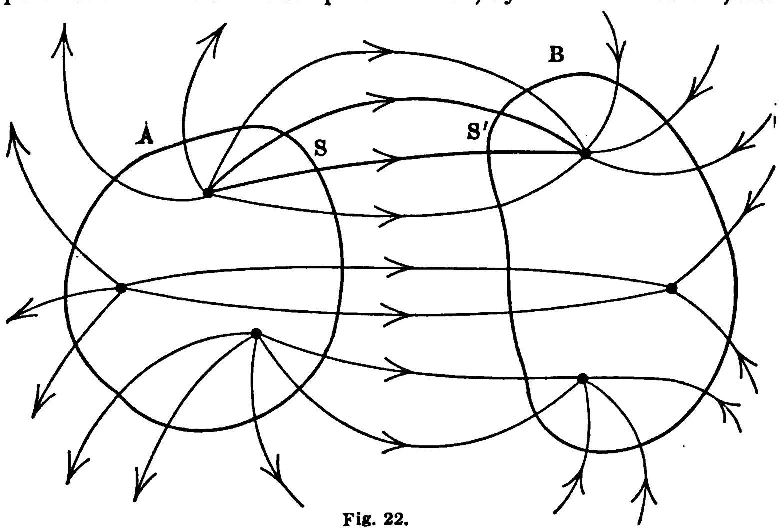


Fig. 22.

algebraic sum of the lines of force coming out from this surface is 4π times the total strength of all the north poles inside this surface. Hence, calling M the total strength of *all* the *north* poles in the field, we have from equation (10a) that

$$\int_A (H \cos a)\, ds = 4 \pi M.$$

Divide this surface A into $4 \pi M$ areas such that the integral of $(H \cos a)\, ds$ over each of these areas is equal to unity, that is, such that the net number of lines of force coming out through each of these areas is unity. Through the perimeters of these areas draw tubes in such a manner that the lateral walls of each tube are tangent at each point to the direction of the *resultant* field intensity at that point. These tubes will fill all space. No tube can cross another since the resultant field intensity can have but a single direction at any one point, and by hypothesis the walls of these tubes are tangent at every point to the resultant field intensity at

that point. Let a second surface B be drawn in the field in such a manner that it encloses all the *south* poles but none of the north poles. Let S be the area cut out of the surface A by any one of these tubes, and S' the area cut out of B by this tube.

Consider the closed surface formed by the lateral walls of this tube and the two areas S and S'. Since there are no poles between S and S', we have from Gauss's Theorem that the net number of lines of force leaving this closed section of the tube is zero. But the net number of lines of force crossing the lateral walls of this tube is zero, since for each area ds in these lateral walls $(H \cos \alpha)\, ds = (H \cos 90°)\, ds = 0$. But this tube is drawn in such a manner that entering it at S the net number of lines of force is unity, since the integral of $(H \cos \alpha)\, ds$ over this surface S is unity, where α is the angle between the direction of the resultant field intensity at ds and the normal drawn from ds *into* the tube. Hence the net number of lines of force leaving the tube at S' must also be unity. Consequently the number of these tubes entering the closed surface B surrounding all the south poles is equal to the net number of lines of force (considered as individual sets of lines) entering this closed surface. But since the total strength of all the south poles inside this surface B is equal to the total strength of all the north poles inside the surface A, the total number of these tubes entering the closed surface B is equal to $4\pi M$. Hence all the tubes leaving the closed surface A surrounding the north poles enter the closed surface B surrounding the south poles. This same relation holds when the two surfaces A and B are drawn infinitely close to the north poles and south poles respectively; consequently 4π of these tubes must originate at every unit north pole in the field and 4π of them end on every unit south pole in the field. Also, the net number of lines of force crossing any surface in the field other than that on which there is a pole, is equal to the number of these tubes crossing this surface.

Hence these tubes are mathematically identical with the lines of force considered as separate sets radiating from or to each individual point-pole in the field, and therefore if we represent each of these tubes by a line coinciding with its axis, we may consider these latter lines as " resultant " lines of force. From equation (10) the number $d\psi$ of these resultant lines of force per unit area crossing any elementary area ds at any point in the field is equal to the component of the resultant field intensity normal to this area, that is

$$H \cos \alpha = \frac{d\psi}{ds} \qquad (10b)$$

where H is the resultant field intensity and α is the angle between the normal to this area and the direction of the resultant field intensity at this point. When the elementary area is taken normal to the direction of these lines the resultant field intensity is equal

to the number of these lines per unit area crossing this elementary area, that is

$$H = \frac{d\psi}{ds_n} \tag{10c}$$

where ds_n represents an elementary area *normal* to the direction of these resultant lines of force.

It should be noted that the idea of lines of force is based upon the idea of field intensity, and consequently the statement that the number of resultant lines of force per unit area perpendicular to their direction is equal to the field intensity *is not a definition.* A definition presupposes a knowledge of *all* the terms in it, and consequently the use of this statement as a definition is a species of "arguing in a circle." The definition of field intensity at any point is the force in dynes due to the agents producing the field that would be exerted upon a unit north point-pole placed at that point (see Article 36). These resultant lines of force are simply a convenient means of representing the resultant magnetic field, these lines being drawn in such a manner that their direction at each point coincides with the direction of the resultant field intensity at that point and their number per unit area at that point crossing a surface perpendicular to their direction is equal to the resultant field intensity at that point.

A rough picture of the direction of the lines of force representing the horizontal component of the resultant field *outside* any

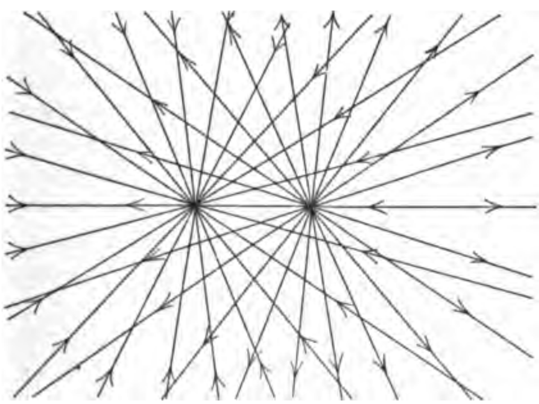


Fig. 23.

number of magnets can readily be obtained by placing the magnets on a table and sprinkling fine iron filings over the table in their

vicinity. When the table is tapped lightly, the filings arrange themselves in the direction of the horizontal component of the field intensity. This is due to the fact that the filings are magnetised by induction and each then acts like a small magnetic needle, setting itself in the direction of the component of the field intensity parallel to the surface of the table. It should be noted, however, that in general the field intensity has a value *inside* the magnets also and that therefore there are lines of force *inside* the magnets as well as outside.

In Fig. 23 are shown the lines of force due to a north point-pole and an equal south point-pole considered as two separate sets of lines, one set radiating out from the north pole and one set radiating into the south pole, and in Fig. 24 are shown the resultant lines of force due to these same two poles. In each case the lines are drawn so that their number per unit area at any point perpendicular to their direction is proportional to the field intensity at this point.

From the definition of the lines of force representing the resultant field, it follows that where the field intensity is great these lines of force will be close together, and where the field intensity is weak these lines of force will be far apart. Again, since 4π of the lines originate from each unit north pole and end on each unit south pole, it follows that these lines will be close together where they enter or leave a surface on which the pole strength per unit area is large, and will be far apart where they enter or leave a surface on which the pole strength per unit area is small. Again, since a surface has two sides, these lines will in general leave *both* sides of a surface on which there is a north pole and enter *both* sides of a surface on which there is a south pole.

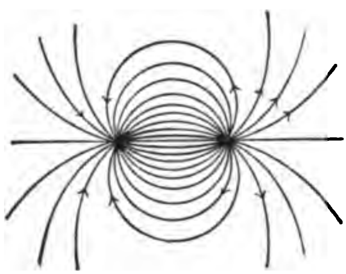


Fig. 24.

46. Intensity of Magnetisation. — When a narrow gap is cut in a magnet, either permanent or induced, it is found that in general poles appear upon the walls of this gap and that these poles are equal and opposite, a north pole appearing on the wall of the gap nearer the south pole of the original magnet, and a south pole on the wall of the gap nearer the north pole of the original magnet.

The strength per unit area of the poles which appear on the wall of such a gap is found to depend upon the direction in which the gap is cut. There is one direction at each point in the magnet for which this pole strength per unit area is a maximum, and when the gap is cut perpendicular to this direction no poles appear on its walls. (In the case of a long bar magnet, a gap cut at its center perpendicular to its axis will have maximum pole strength per unit area, while on a gap cut at this point parallel to its axis there will be no poles formed.)

A magnetised body may then be considered as made up of magnetic filaments such that were the lateral walls of any one of these filaments separated from the rest of the magnet by a narrow air gap, no poles would be formed on these lateral walls. The two ends of any such filament where it terminates in the surface of the magnetised body must then have equal and opposite magnetic poles; if the filament is cut transversely by a narrow air gap at any point, equal and opposite poles will be formed at the two walls of this air gap, and these poles will in turn be numerically equal to the poles at the ends of the filament in the original surface of the magnetised body. (For, since each filament is a magnet, it must have equal and opposite poles, and by definition each filament has no poles on its lateral walls.) Such a filament is considered as existing only in a magnetised body; the filament is broken by the transverse air gap just as a string is broken when it is cut in two.

Consider such a filament cut at any point by an air gap perpendicular to its axis, and let ds_n be its cross section at this point, and let dm be the pole strength of the north pole formed where it ends in the gap. Then the pole strength per unit area of this pole is

$$\sigma = \frac{dm}{ds_n}$$

or *vice versa*, the strength of the pole dm formed where the filament is cut by the gap is

$$dm = \sigma_n \, ds_n$$

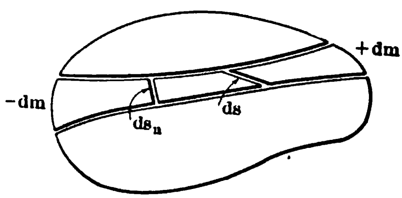


Fig. 25.

If the filament is broken at any other point, poles of exactly the same strength will be formed on the broken ends, a north pole on the end nearer to the original south pole of the magnetised body and a south pole on the

opposite end. Also, the strength of the poles where this filament ends in the original surface of the magnetised body will be numerically equal to dm. Consequently where the cross section ds_n of the filament is great the strength per unit area of the pole which would be formed were it broken in two is small, and where the cross section is small the pole strength per unit area is large. The strength per unit area of the pole which would be formed on the walls of a gap cut at any point in a magnetised body perpendicular to the direction of the magnetic filaments of which the body may be considered as made up is defined as the *intensity of magnetisation** of the body at this point, and may be represented by the symbol J. That is,

$$J = \frac{dm}{ds_n} = \sigma_n \tag{11}$$

where ds_n is the cross section of the filament at the point under consideration and dm is the strength of the north pole which would be formed on one wall of a narrow gap coinciding with the cross section ds_n. The direction of the intensity of magnetisation J is chosen arbitrarily as the direction of the magnetic filament at this point and the *positive sense* of J is chosen as the sense of the line drawn *into* the gap *from* the wall of the gap on which the *north* pole is formed.

When a magnetic filament is cut by a gap ds which is not at right angles to the axis of the filament, the numerical value of the strength of the pole formed on either wall of the gap must be equal to the strength of the pole which would be formed on a gap ds_n cut normal to the filament, since the pole formed on either gap must be equal to the strength of the pole at either end of the filament in the original surface of the magnetised body. Let σ be the pole strength per unit area of the north-pole end of the filament in the surface of the gap, let α be the angle between the direction of this filament and the normal drawn *outward* into the gap from

*Intensity of magnetisation at any point may also be defined as the magnetic moment per unit volume of an infinitesimal length of the magnetic filament passing through that point. For, calling ds_n the cross section of the filament, σ_n the numerical value of the pole strength per unit area at each end of the infinitesimal length dl of the filament; the magnetic moment of this element of the filament is $(\sigma_n \, ds_n) \, dl$ and the volume is $ds_n dl$, whence the magnetic moment per unit volume is σ_n; and therefore this definition is equivalent to that given above.

this north pole, and let ds_n be the projection of ds on a plane normal to the direction of the filament. Then from (11)

$$dm = \sigma\, ds = J ds_n$$

But

$$ds_n = ds \cos a$$

whence

$$\sigma = J \cos a \qquad (11a)$$

That is, the pole strength per unit area which would be formed on the wall of a narrow gap cut in any direction at any point in a magnet is equal to the component of the intensity of magnetisation in the direction of the normal *drawn outward* from this wall into the gap.

Intensity of magnetisation may be represented by lines just as magnetic field intensity may be represented by lines. This is done by choosing arbitrarily the size of a unit magnetic filament and representing each unit filament by a line coinciding with its axis. As the unit magnetic filament is taken a filament such that were it cut by a narrow air gap at any point, the strength of the pole formed on either wall of the gap would be equal to $\frac{1}{4\pi}$. The line representing such a filament is called a *line of magnetisation;* the direction of this line coincides with the direction of the intensity of magnetisation.

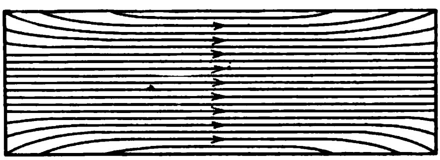
S  N

Fig. 26. Lines of Magnetisation in a Bar Magnet.

Hence from every unit south pole in the surface of a magnetised body 4π lines of magnetisation originate, these lines run through the magnetised body to the surface over which the north pole is distributed, 4π of them ending in every unit north pole in the surface of the magnet.

The reason for introducing the factor $\frac{1}{4\pi}$ is to have number of these filaments *leaving* a south pole equal to the lines of force *entering* that pole and the number of these filaments *entering* a north pole equal to the lines of force *leaving* that pole. It should be

noted that these filaments are confined entirely to magnets or magnetised bodies, while lines of force in general exist in all the space surrounding a magnetic pole, whether this space be occupied by a non-magnetic or by a magnetic body.

The relation between the number of lines of magnetisation dN crossing *any* elementary surface ds and the intensity of magnetisation at ds is given by the formula

$$dN = 4\pi (J \cos \alpha)\, ds \tag{12}$$

where α is the angle between the direction of these lines at ds and the direction of the normal to ds. $J \cos \alpha$ is the *component* of the intensity of magnetisation normal to the surface ds. Compare with the mathematical expression (10) for the number of lines of force crossing an area.

Since in the case of a long bar magnet the poles are confined almost entirely to its ends, the lines of magnetisation inside the magnet near its center must be parallel to the sides of the magnet. When such a magnet is cut in two by a plane surface perpendicular to its axis, the magnetic poles which appear on the walls of the gap thus formed must then, from equation (11), have a strength per unit area equal to the intensity of magnetisation at this surface. (This is strictly true only in case the gap between the two parts of the magnet is of infinitesimal width.)

This fact suggests a method for the experimental determination of the intensity of magnetisation of a bar which can be separated in two parts. For, by equation (6), the force required to separate the two equal and opposite poles which appear on the walls of the gap formed by separating the two parts of the bar is

$$F = 2\pi \sigma^2 S$$

where S is the cross section of the bar and σ the strength of these poles per unit area, which may be assumed constant over each wall of the gap. Hence in this case the intensity of magnetisation is

$$J = \sigma = \sqrt{\frac{F}{2\pi S}} \tag{13}$$

and both F and S are readily measured. In employing this method certain corrections have to be made for the effect of the action of other forces on the poles which are separated. A description of the method and apparatus will be found in Foster's Pocket Book, p. 94.

(In the above discussion the matter forming a magnet is con-

sidered as absolutely continuous, and no hypothesis is made as to its molecular structure, just as in the ordinary theory of the elastic properties of matter a beam or column is considered as made up of continuous fibers and no account is taken of the molecular structure of these fibers.

The modern theory of the molecular structure of a magnet assumes that each molecule of the magnet contains one or more electrically charged particles, which by their motion give rise to lines of magnetic force which form closed loops threading the molecules of the magnet. Inside the magnet these lines of force are in the direction from the south pole of the magnet to its north pole and outside in the direction from its north pole to its south; the pole of a magnet is then simply an *apparent* property possessed by the surface of the magnet where it is cut by these lines. These closed lines of force are then identical with what will be defined presently as lines of induction. This hypothesis, while probably correct, can be fully understood only after one has become thoroughly acquainted not only with the facts concerning magnetism, but also with the phenomena of electric currents and electrostatics. We shall therefore still continue to consider a magnetic pole as an actual entity which has the properties which have been assigned to it, and to avoid confusion, shall distinguish carefully between lines of force, lines of magnetisation and lines of induction.)

47. Lines of Magnetic Induction.— Flux of Induction. — We have seen that a magnet may be considered as made up of a number of magnetic filaments, or lines of magnetisation, at the ends of which are located the poles of the magnet, and that these poles in turn give rise to lines of force equal in number to the number of lines of magnetisation in the magnet. The lines of magnetisation are confined entirely to the substance of the magnet and are considered as originating at its south pole and running through the magnet to its north pole. Lines of force exist both in the magnet and in the surrounding space and are considered as originating at the north pole of the magnet and running through both the medium surrounding the magnet and the magnet itself to its south pole. In the substance of the magnet there are therefore both lines of magnetisation and lines of force, but the lines of force in a magnet *due solely to the poles of this magnet* are in the opposite direction to the lines of magnetisation. In the case of an induced magnet, *e.g.*, a piece of soft iron in a magnetic field produced by some other agent, the *resultant* lines of force are in general in the same direction as the lines of magnetisation. In any case the *algebraic* sum of the lines of magnetisation and the lines of force crossing any surface in space, whether this surface be in a magnetic

body or a non-magnetic body or in free space, is defined as the number of lines of *magnetic induction* crossing that surface, or the *flux of magnetic induction* across that surface. The unit of flux of magnetic induction is called the *maxwell*, that is, 1 maxwell = 1 *c. g. s.* line of magnetic induction.

From equations (10) and (12) the mathematical expression for the number of lines of magnetic induction crossing any elementary area ds is

$$d\phi = (4\pi J \cos \alpha_1 + H \cos \alpha_2)\, ds \qquad (14)$$

where J and H are the intensity of magnetisation and the field intensity respectively at ds and α_1 and α_2 are the angles between the normal drawn to ds and the directions of the intensity of magnetisation and field intensity respectively. The direction of the lines of induction through the elementary area ds is taken as the direction of the vector which is equal to the vector sum of $4\pi J$ and H. As a rule, in all practical applications when the field intensity and intensity of magnetisation are due to the same cause (*e. g.*, when a piece of soft iron is magnetised by the action of an electric current), these two quantities are in the same direction, and this expression may then be written

$$d\phi = (4\pi J + H) \cos \alpha .\, ds. \qquad (14a)$$

where α is the common angle made by J and H with the normal to ds; in this case the lines of force, lines of magnetisation, and lines of induction all coincide in direction.

From the definition of lines of magnetisation, there can be no lines of magnetisation in air or in any other non-magnetic substance. Hence in air or in any other non-magnetic substance the lines of force and the lines of induction are identical, but *this is never the case in a magnetic or diamagnetic substance*, for when such a substance is placed in a magnetic field it becomes magnetised by induction (see Article 32) and consequently lines of magnetisation, as well as lines of force, are produced in the substance.

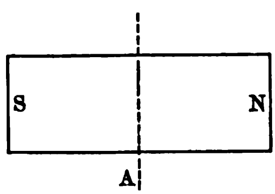


Fig. 27.

Consider first the lines of magnetisation in a single magnet and the lines of force due to its poles. Let N be the total number of lines of *magnetisation* in this magnet, ψ_0 the number of lines of *force outside* the magnet, and ψ_1 the number of lines of *force inside* the magnet. Outside the magnet the total number of lines of in-

duction is then simply ψ_0. Inside the magnet, across the section A taken perpendicular to its axis at its middle point, pass all the lines of magnetisation N and the ψ_1 lines of force, the former in the direction S to N and the latter in the direction from N to S. Hence the total number of lines of induction *through the magnet* across this area A is

$$\phi = N - \psi_1$$

But since the *total* number of lines of magnetisation is equal to the *total* number of lines of force, we also have

$$N = \psi_0 + \psi_1$$

whence

$$\psi_0 \mp N - \psi_1.$$

But ψ_0 is also equal to the total number of *lines of induction* outside the magnet. Hence the total number of lines of induction passing *through* a permanent magnet from its *south* to its *north* pole is equal to the total number of lines of induction passing back *outside* the magnet from its *north* to its *south* pole. Therefore each line of induction must be a closed loop, part of which lies inside the magnet and part outside. The magnet may then be looked

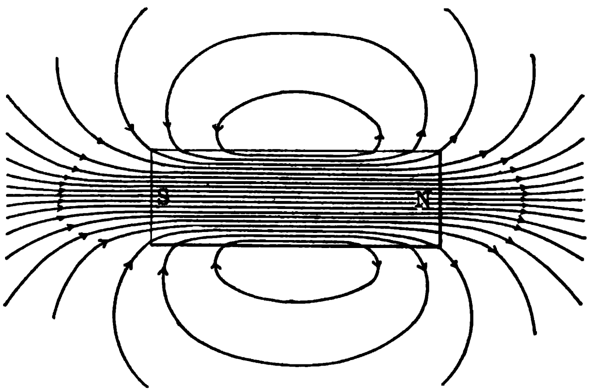


Fig. 28. Lines of Magnetic Induction due to a Bar Magnet.

upon as a sheath which binds these lines of induction closely together (Fig. 28); these lines spread out from one end of the sheath, bend around and re-enter the sheath at the other end. Fig. 28 should be compared with Fig. 29 which shows the lines of *force* due to a single magnet. It should be noted that since some of the lines of magnetisation end in the lateral walls of the magnet, part of these lines of induction pass *through* its lateral walls. In the case of a long slim magnet, however, practically all the lines of induc-

tion pass through its ends. In general, wherever there is a magnetic pole on the surface of a magnetic substance there must also

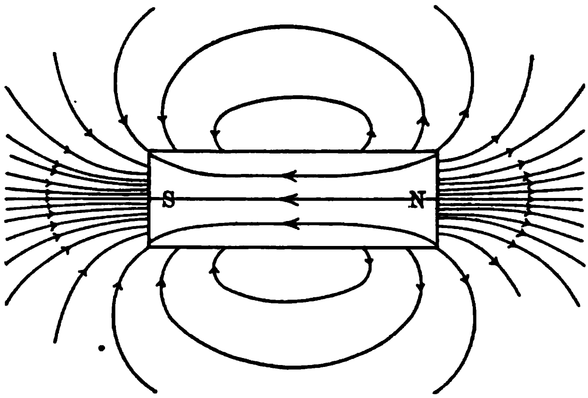


Fig. 29. Magnetic Lines of Force due to a Bar Magnet.

be a line of induction passing through this surface; or, *vice versa*, wherever a line of induction passes through the surface of a magnetic substance there must be a pole on its surface.

Since each line of induction due to a single magnet forms a closed loop it follows that such a line of induction will always cut a closed surface an even number of times, as many times in the direction from the outside to the inside as in the direction from the inside to the outside. Hence, adopting the same convention as

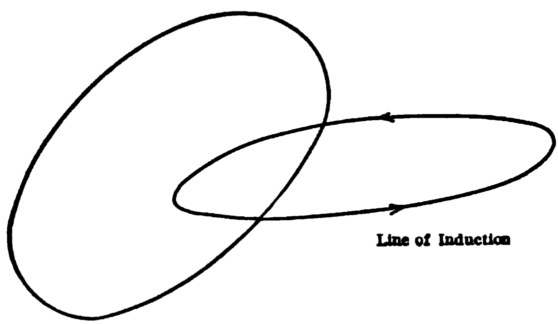


Fig. 30.

in the case of lines of force, namely, that a line entering a surface is equivalent to a *negative* line leaving that surface, it follows that the algebraic sum of the lines of induction outward across any closed surface is always zero, even though this surface encloses the pole of

a magnet. The difference in this respect between lines of force and lines of induction should be carefully noted; 1, lines of force *end* on magnetic poles, while lines of induction *pass through* magnetic poles; 2, the lines of force due to a magnet therefore have *ends*, while the lines of induction are *closed curves;* 3, the algebraic sum of the lines of force outward across a closed surface is equal to 4π times the algebraic sum of the magnetic poles enclosed by this surface, while the algebraic sum of the lines of induction across any closed surface is always zero.

48. Flux Density. — We have seen that the number of lines of force per unit area crossing any elementary surface ds is equal to the component of the intensity of the magnetic field normal to this area (equation 10). We shall also find that the number of lines of *induction* per unit area crossing any elementary surface plays a very important role in the theory of magnetism and electromagnetism. When the surface is taken normal to the lines of induction through it, the number of these lines of induction per unit area crossing this surface is called the *density of the lines of induction* at the point occupied by this elementary surface or simply the *flux density* at that point, and the direction of this flux density is defined as the direction of the lines of induction at this surface. Flux density is therefore a vector quantity, since it has both magnitude and direction. The symbol usually employed for flux density is the capital letter B.

The mathematical expression for the flux density in terms of the intensity of magnetisation J and the field intensity H when both J and H are in the same direction is therefore

$$B = 4\pi J + H \tag{15}$$

See equation (14*a*).

When J and H are not in the same direction $4\pi J$ and H must be added vectorially; see equation (14). This can be done by resolving J and H along two mutually perpendicular axes, adding the components of $4\pi J$ and H along these respective axes, and taking the square root of the sum of the squares; see Article 9.

The unit of flux density on the *c. g. s.* system is one line of induction per square centimeter, or one maxwell per square centimeter; this unit is called the *gauss*. Flux density may also be expressed as so many *c. g. s.* lines per square inch, or so many thousands of *c. g. s.* lines per square inch. A thousand lines is

called a kilo-line. Hence the following relations between the various units of flux density:

1 gauss	=1 maxwell per square centimeter
1 gauss	=1 *c. g. s.* line per square centimeter
1 line per square inch	=0.15500 gauss
1 kilo-line per square inch	=155.00 gausses

From the definition of flux density, the number $d\phi$ of lines of induction crossing any elementary surface ds may be expressed in terms of the flux density by the formula

$$d\phi = (B \cos \alpha)\, ds \qquad (16)$$

where B is the flux density at ds and α the angle between the direction of the lines of induction through ds and the normal to ds; this follows immediately from equation (14). Compare this with the expression for the number of lines of force across any elementary surface ds (equation 10).

The above definition of flux density and equations (14) and (15) are applicable to any magnetic field, no matter how this field may be produced, whether by a single permanent magnet, by any number of magnets, or by an electric current. Moreover, just as the separate sets of lines of *force* due to any number of magnets may be represented by a single set of "resultant" lines of force, so the separate sets of lines of *induction* due to any number of magnets may be represented by a single set of "resultant" lines of induction, and these resultant lines of induction always form closed loops, just as the lines of induction due to a single magnet are closed loops.

For, let ds be any elementary surface in the field, B_1 B_2 etc., the flux densities at ds due to the respective magnets, and α_1, α_2, etc., the angles between the normal to ds and the directions of the lines of induction through ds due to the respective magnets. Then the number of lines of induction across ds due to the first magnet is $(B_1 \cos \alpha_1)\, ds$; the number of lines of induction across ds in the same direction due to the second magnet is $(B_2 \cos \alpha_2)\, ds$; etc. Hence the net number of lines of induction across ds in the same sense due to *all* the magnets is

$$d\phi = (B_1 \cos \alpha_1 + B_2 \cos \alpha_2 + \text{——})\, ds = (B \cos \alpha)\, ds$$

and the total number of lines of induction across any finite surface is then

$$\phi = \int_S (B_1 \cos \alpha_1)\, ds + \int_S (B_2 \cos \alpha_2)\, ds + \text{——} = \int_S (B \cos \alpha)\, ds$$

where B is the resultant or vector sum of B_1 B_2 etc., and α is the angle between the normal to ds and the direction of the resultant

flux density B. Hence we may look upon the independent sets of lines of induction $\int_S (B_1 \cos \alpha_1)\, ds$, $\int_S (B_2 \cos \alpha_2)\, ds$, etc., as combining and forming one set of lines equal in number to $\int_S (B \cos \alpha)\, ds$, the direction of each of these lines at each point in its path coinciding with the direction of the *resultant* flux density at that point.

Since the net number of lines of induction due to a single magnet outward across any closed surface is zero (Article 47, last paragraph), the algebraic sum of the lines of induction due to any number of magnets, outward across any closed surface, must also be zero, that is

$$\int_{|S|} (B \cos \alpha)\, ds = \int_{|S|} (B_1 \cos \alpha_1)\, ds + \int_{|S|} (B_2 \cos \alpha_2)\, ds + \text{———} = 0$$

since each term of the middle member of this equation is zero. Hence the lines of induction due to any number of magnets, whether these magnets be permanent or induced, must form closed loops, and therefore the number of these lines coming up to any surface on one side must always be equal to the number of these lines which leave the other side of this surface. When we come to the study of electric currents we shall also see that the lines of induction produced by an electric current are also closed loops. *Hence a line of induction always forms a closed loop, no matter how it may be produced.*

49. The Normal Components of the Flux Density on the Two Sides of any Surface are Equal. — The fact that the number of lines of induction coming up to one side of a surface must equal the

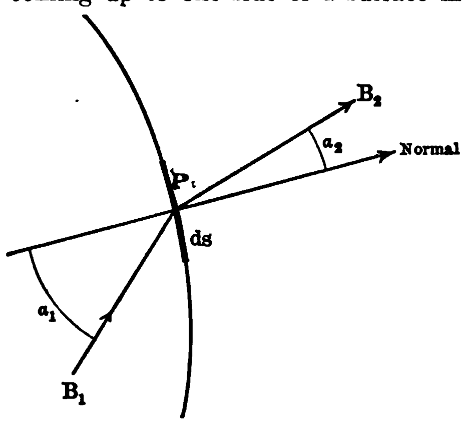


Fig. 31.

number of these lines which leave the other side, is a very important one in the theory of magnetism, and may be expressed mathematically as follows. Let B_1 and B_2 be the flux densities on the

two sides of any surface directly opposite any point P in this surface, and let α_1 and α_2 be the angles between the direction of the normal drawn *through* the surface at this point and the directions of these flux densities. Let ds be any elementary area of the surface at P. Then the number of lines of induction coming up to ds on one side is $(B_1 \cos \alpha_1)\, ds$, and the number of lines of induction leaving ds on the other side is $(B_2 \cos \alpha_2)\, ds$. Therefore, since the number of lines of induction coming up to ds must equal the number leaving ds,

$$(B_1 \cos \alpha_1)\, ds = (B_2 \cos \alpha_2)\, ds$$

or

$$B_1 \cos \alpha_1 = B_2 \cos \alpha_2 \tag{17}$$

That is, the *normal components of the flux densities on the two sides of the surface are equal*, and this is true even though the surface is the seat of a magnetic pole. When there is no pole at the surface, the tangential components of the flux densities ($B_1 \sin \alpha_1$ and $B_2 \sin \alpha_2$) are also equal, but when there is a pole at the surface, the tangential components will not in general be equal. In the latter case the lines of induction make an abrupt change in direction. See Article 54.

50. The Tangential Components of the Field Intensity on the Two Sides of any Surface are Equal. — While lines of induction are always continuous lines forming closed loops, and therefore always

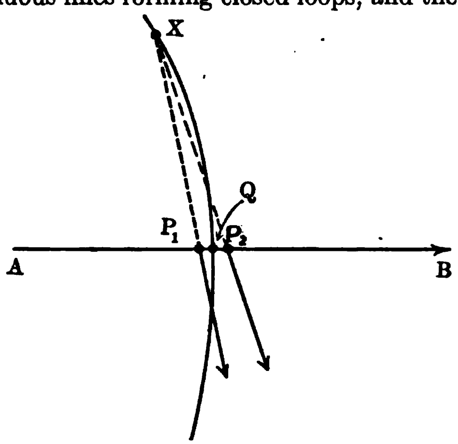


Fig. 32.

pass *through* any surface in their path, lines of force *originate at or end on magnetic poles*. Hence the normal components of the field intensities on the two sides of a surface on which there are magnetic poles are *not* equal; the *tangential* components of these field intensities are however *always* equal, even though the surface be magnetised. This may be proved as follows. Let AB be the normal through the surface at any point Q and let P_1 be a point on

this normal just outside the surface on one side and P_2 a point on this normal just outside the surface on the other side, and let QP_1 and QP_2 be equal. Then the field intensities at P_1 and P_2 due to any pole at any point X other than Q will differ in magnitude and direction by an amount depending upon the distance QP_1 and QP_2 and in the limit, when P_1 and P_2 coincide with Q, these field intensities due to the pole at X will be exactly equal both in magnitude and direction. Hence both the normal and the tangential components of the field intensities at P_1 and P_2 due to a pole at any other point than Q will be respectively equal when P_1 and P_2 coincide with or are infinitely close to Q. If, however, there is a magnetic pole at Q, this pole will produce equal and opposite field intensities at P_1 and P_2, and these intensities will still be equal and opposite when P_1 and P_2 are infinitely close to the surface. But since by hypothesis P_1 and P_2 are on the normal to the surface at Q, the field intensities at these two points due to the pole at Q will also be *normal* to this surface and therefore have no tangential components. Hence the pole at Q has no effect upon the tangential components at P_1 and P_2. Therefore, since the tangential components at P_1 and P_2 due to all the other poles in the field are also equal, the resultant tangential components at P_1 and P_2 due to *all* the poles in the field are equal. The resultant normal components of the field intensities at P_1 and P_2 are however *not* equal when there is a pole at Q, since the field intensity at P_1 due to the pole at Q is opposite to the field intensity at P_2 due to this pole.

Calling α_1 and α_2 the angles between normal drawn through any point of a surface and the directions of the field intensities H_1 and H_2 on the two sides of this surface, we then have that

$$H_1 \sin \alpha_1 = H_2 \sin \alpha_2 \qquad (18)$$

51. Conditions which must be Satisfied at every Surface in a Magnetic Field. — The above deductions concerning lines of magnetic induction and lines of force hold whether the magnets producing the field are permanent or induced. The two surface conditions just deduced hold for every surface in the field, and consequently must hold at the surface of any magnetic body placed in the field. By taking this fact into account one can calculate in certain simple cases the exact distribution of the magnetic poles induced on the surface of a magnetic body when placed in a magnetic field, and also the distribution of the lines of force and the lines of induction. (See J. J. Thomson, El. of Elec. & Mag. p. 257ff.) These surface conditions may be summed up:

1. The normal components of the flux densities on the two sides of any surface are always equal.

2. The tangential components of the field intensities on the two sides of any surface are always equal.

52. Induced Magnetisation.— We have seen that when a magnetic body which itself is not a magnet is placed in a magnetic field, this body becomes a magnet; this phenomenon is described by saying that the body becomes magnetised by induction. To fix ideas, let this field be that in the vicinity of the north pole of a bar magnet, and let the magnetic body be a soft iron

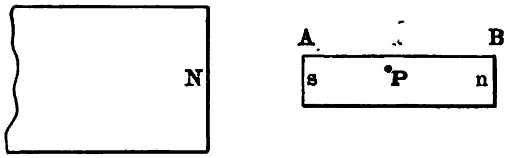


Fig. 33.

rod placed as shown. As we have already seen, the end A of this rod becomes a south pole and the end B a north pole. Consequently inside the rod AB there will be (1) a field of force due to the original field of the bar magnet in the direction from A to B and (2) a field of force due to the magnetic poles "induced" on the rod, which field will be in the direction from B to A. Let H be the numerical value of field intensity at any point P inside the rod due to the original magnetic field, H' the numerical value of the field intensity at this point due to the poles induced on the rod, and J the numerical value of the intensity of magnetisation of the rod at this point. The induced intensity of magnetisation J may be considered as *produced by* the original field H. The intensity H′ which is due to the induced poles and which therefore is approximately in the opposite direction to the original field H, would of itself tend to magnetise a body in the opposite direction, and is therefore called the "demagnetising force" due to the ends of the rod. Experiment shows that this demagnetising force H' is always less than the magnetising force H but is *not negligible* unless the field in which the rod is placed is uniform and the rod itself is very long. The resultant field intensity at P will then be the vector difference

$$H_r = \overline{H - H'}$$

When H and H' are in exactly opposite directions the resultant field intensity H_r is the arithmetical difference between H and H' or

$$H_r = H - H'$$

This condition is approximately realized in the case of a long,

slim bar placed in a uniform magnetic field parallel to the direction of the field. Experiment also shows that when a body which of itself is not a magnet (*e.g.*, the soft iron rod we are considering) is placed in a magnetic field, the intensity of magnetisation produced at any point in the body is in general in the direction of the resultant field intensity. (Certain exceptions will be noted later, Article 57). Consequently, across any elementary surface ds drawn normal to the direction of the resultant field intensity there will be $H_r\ ds$ lines of force and $4\ \pi\ J\ ds$ lines of magnetisation, both as a rule in the *same* direction, namely from A to B (see equations 10 and 12). Hence the total number of lines of induction across this area will be (see equation 14*a*)

$$d\phi = (H_r + 4\ \pi\ J)\ ds$$

and therefore the flux density at the point P, *i.e.*, the number of lines of induction per unit area normal to the direction of these lines, will be

$$B = \frac{d\phi}{ds} = H_r + 4\ \pi\ J \qquad (19)$$

53. Magnetic Permeability. — The resultant intensity of magnetisation J produced in a piece of iron or steel when it is placed in a magnetic field is in general many times greater than the original intensity of the field, and therefore the number of lines of *induction* through the space occupied by this piece of iron or steel is greatly increased by its presence,* although the number of lines of *force* through this space is in general decreased (the latter is *always* true except when there are no poles produced on the surface of the body which is magnetised by induction — we shall see later when we come to the study of electric currents how a body may be magnetised without producing any poles on its surface).

Hence iron or steel, or in fact any magnetic substance, is said to be more "permeable" to lines of induction than a non-magnetic body (diamagnetic bodies are less permeable), and the ratio

*The fact that when a magnetic body is placed in a magnetic field the number of lines of induction through the space occupied by the body is increased, is frequently described by saying that the lines of induction tend to crowd into a magnetic body when placed in a magnetic field. In general, however, the presence of such a substance in a magnetic field not only causes the original lines of induction to crowd into the substance, but also gives rise to an additional number of these lines. This is particularly true in the case of an iron rod placed in the magnetic field produced by an electric current flowing in a coil of wire.

of the resultant flux density at any point of the body to the resultant field intensity is called the *permeability* of the body at that point, and is usually represented by the symbol μ. That is

$$\mu = \frac{B}{H_r} \tag{20}$$

where H_r is the *resultant* field intensity, which in case just considered is equal to $\overline{H - H'}$. Since there can be no lines of magnetisation in a non-magnetic body, the flux density and field intensity in such body are numerically equal, and therefore for all non-magnetic bodies $\mu = 1$. For magnetic bodies μ is always greater than 1 and for diamagnetic bodies less than 1. Strictly speaking, the permeability is unity only for air, since all bodies are slightly magnetic or diamagnetic with respect to air. However, μ is practically unity for all substances other than iron, steel, nickel, cobalt and bismuth; for the last μ is less than unity and for the rest greater than unity.

The permeability of any of these substances is not a constant, but depends upon the intensity of the resultant magnetic field, and also upon its previous history, whether it is already a magnet before being placed in the field of force and upon how the field of force inducing the magnetisation is established. In fact, the permeability may be *negative*, that is, the resultant flux density may be in the opposite direction to the resultant field intensity. This is always true of a single permanent magnet by itself. In practice, however, the permeability of a body is taken to mean the ratio of the flux density to the resultant field intensity *when the body is originally unmagnetised and then placed in a field which is continuously increased from zero to its final value.*

54. Refraction of the Lines of Induction at the Surface of Separation of Two Bodies of Different Permeabilities. — It can readily be shown, by making use of the surface conditions given in Article 51, that wherever a line of induction crosses the surface of separation between two bodies of different permeabilities, this line is refracted *toward* the normal at this surface in the body of *lesser* permeability. Let the permeabilities of the two bodies directly opposite any point Q in the surface separating them be μ_1 and μ_2 respectively : let H_1 and H_2 be the field intensities in the two bodies respectively at points infinitely close to Q on the two sides of the surface; let B_1 and B_2 be the flux densities at these two points and α_1 and α_2 the angles between the normal drawn through the surface at Q and the directions on the two sides of the surface of the line of induction through Q. The line of force through

Q will coincide in direction with the line of induction (provided the bodies are not crystals and are magnetised solely by induction), hence α_1 and α_2 will also be the angles between the normal at Q and the directions of the lines of force on the two sides of the surface. Hence from the surface conditions given in Article 51,

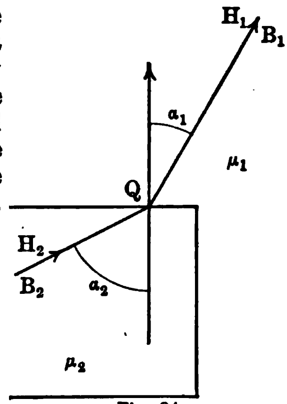


Fig. 34.

$$B_1 \cos \alpha_1 = B_2 \cos \alpha_2$$
$$H_1 \sin \alpha_1 = H_2 \sin \alpha_2.$$

Hence

$$\frac{H_1}{B_1} \tan \alpha_1 = \frac{H_2}{B_2} \tan \alpha_2.$$

But $B_1 = \mu_1 H_1$ and $B_2 = \mu_2 H_2$. Whence

$$\tan \alpha_1 = \frac{\mu_1}{\mu_2} \tan \alpha_2. \tag{21}$$

Hence if μ_1 is less than μ_2, then α_1 is less than α_2; that is, the line of induction is bent toward the normal in the body of lesser permeability. For example, the permeability of soft iron is about 3000 under ordinary conditions. Hence a line of induction coming up to the surface inside the iron at an angle of 45° say, comes out into the air at an angle of $\alpha = \tan^{-1} \frac{1}{3000}$ with the normal, that is, comes out into the air practically at right angles to the surface. Therefore, when a piece of soft iron is placed in a magnetic field in air practically all the lines of induction which pass through it enter and leave its surface approximately at right angles to the surface.

55. Value of the Pole Strength per Unit Area Induced on the Surface of Separation of Two Bodies of Different Permeabilities.— From the surface condition that the number of lines of induction coming up to the surface is equal to the number of lines leaving the surface, can also be deduced an expression for the net pole strength per unit area induced on the surface of separation between the two bodies. Let ds (Fig. 35) be an elementary area in this surface at Q and draw about ds a closed cylinder having the cross section ds and its ends infinitely close to ds. Applying Gauss's Theorem (Art. 44) to this closed cylinder, we have

$$4\pi\sigma\, ds = H_1 \cos \alpha_1\, ds - H_2 \cos \alpha_2\, ds$$

whence

$$\sigma = \frac{H_1}{4\pi}\left(\cos \alpha_1 - \frac{H_2}{H_1} \cos \alpha_2\right)$$

where σ is the pole strength per unit area on ds. But since the number of lines of induction coming up to ds is equal to the number of lines of induction leaving ds, we also have

$$B_1 \cos \alpha_1 = B_2 \cos \alpha_2$$

or, since $B_1 = \mu_1 H_1$ and $B_2 = \mu_2 H_2$

$$\frac{H_2}{H_1} \cos \alpha_2 = \frac{\mu_1}{\mu_2} \cos \alpha_1$$

which substituted in the above equation, gives

$$\sigma = \frac{H_1 \cos \alpha_1}{4\pi} \left[1 - \frac{\mu_1}{\mu_2}\right] \qquad (22)$$

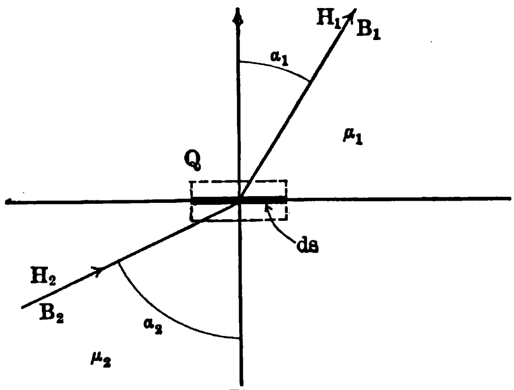


Fig. 35

Hence if the line of induction at Q passes from a medium of low to one of high permeability (*i.e.*, $\mu_1 > \mu_2$), the pole induced on the surface is negative, while if the line of induction passes from a medium of high to one of low permeability (*i.e.*, $\mu_1 < \mu_2$) the pole induced on the surface is positive. This explains the fact, noted in Article 29, that the direction of the force produced on an originally unmagnetised body placed near a magnet depends upon the nature of the medium between the body and the magnet, and that whether this force is an attraction or a repulsion depends upon the *relative* permeabilities of the body and the surrounding medium. When μ_1 and μ_2 are both different from unity there will be a pole induced on each of the substances in contact; equation (22) gives the *algebraic sum* of these pole strengths per unit area.

56. Field Intensity at any Point in Magnetic Medium of Constant Permeability Completely Surrounding a Point-Pole and Filling all Space.—An important relation in the theory of magnetism is that the *resultant* field intensity at any point in a magnetic medium of constant permeability μ completely surrounding a point-pole of strength m is $H_o = \frac{m}{\mu r^2}$, where r is the distance of the point from the pole. This is equivalent to stating that the pole m induces on the surface of the medium in contact with it a pole of opposite sign, the numerical strength of which is equal to $\left(1 - \frac{1}{\mu}\right) m$, for then the *resultant* field intensity will be

$$H_o = \frac{m}{r^2} - \left(1 - \frac{1}{\mu}\right) \frac{m}{r^2} = \frac{m}{\mu r^2} \qquad (23)$$

To prove this, consider a magnetic filament of infinite length and infinitesimal cross section, and let the poles of this filament have the constant strengths m and $-m$. We may imagine the pole $-m$ at infinity and therefore producing no field intensity at any point in the vicinity of m. Again, since the filament is assumed to have infinitesimal cross section, we may neglect the non-uniformity produced by it in the medium surrounding m. The lines of force due to the pole m will therefore be equally spaced lines radiating out from m in all directions. When the pole is surrounded by a non-magnetic medium, *i.e.*, one having unit permeability, the lines of induction in the surrounding medium will coincide with these lines of force; the lines of induction come up to the pole through the magnetic filament, as lines of *magnetisation*, and then radiate out from the pole, as lines of *force*, into the surrounding medium. Since the pole m is constant by hypothesis, the intensity of magnetisation of the filament must also be constant; whence the number of lines of magnetisation, $4\pi m$, in the filament is constant, and since the filament has an infinitesimal cross section, the lines of force inside the filament will be negligible in comparison with the number of lines of magnetisation. Hence the number of lines of induction coming up to the pole m through the filament must also be constant and equal to $4\pi m$; consequently the number of lines of induction radiating out into the surrounding medium is $4\pi m$ independent of the nature of the medium.

When the pole is surrounded by a non-magnetic medium, the field intensity at any point is $H=\frac{m}{r^2}$, and since the medium is non-magnetic the flux density is also $B=\frac{m}{r^2}$. Since the lines of induction are the same whether the surrounding medium is magnetic or non-magnetic, the flux density at any point in any medium whatever completely surrounding a point-pole of strength m is $B=\frac{m}{r^2}$. Consequently, when the medium has a permeability μ, the field intensity at any point in it due to the pole m must be $H=\frac{B}{\mu}=\frac{m}{\mu r^2}$. The number of lines of force outward across the surface of any sphere surrounding m is therefore $\frac{4\pi m}{\mu}$; but by Gauss's Theorem this must be equal to 4π times the algebraic sum of the poles inside this sphere; hence calling m_i the value of the strength of the pole induced by m on the surface of the medium in contact with it, we have, $4\pi(m+m_i)=\frac{4\pi m}{\mu}$ and therefore $m_i=-\left(1-\frac{1}{\mu}\right)m$.

From the fact that the resultant field intensity at any point in a

medium of constant permeability completely surrounding a point-pole of strength m is $\frac{m}{\mu r^2}$, it follows that the resultant force of repulsion on a second point-pole of strength m' placed in the medium at any point is $f = \frac{m m'}{\mu r^2}$, where r is the distance between the poles and μ is the permeability of the medium. It should be noted that this is the *resultant* force acting on m' due to both the pole m and the pole $-\left(1 - \frac{1}{\mu}\right) m$ induced by m on the surface of the medium in contact with it. The force due to m alone is $\frac{m m'}{r^2}$ and is *independent* of the nature of the surrounding medium.

The above deductions hold only when the poles are *completely* surrounded by the medium of constant permeability μ. In any actual case this can never be true, since a magnetic pole can exist only where there is a surface of separation between two substances of *different* permeabilities.

57. Magnetic Hysteresis. — As noted in Article 53, the flux density produced in a given piece of iron or other magnetic substance by a given field intensity depends upon the previous history of the sample. To make this fact clearer, consider the special case of an originally unmagnetised rod of soft iron placed in a uniform magnetic field with its axis parallel to the direction of the field, and let this field be gradually increased from zero up to some maximum value H_m and then decreased to zero, then increased in this reversed direction to an equal negative maximum value $-H_m$, decreased to zero again and then again increased to the same maximum value H_m in the original direction. It is found by experiment that the relation between the flux density and the field intensity during the various steps in this process may be represented by a curve of the form shown in Fig. 36. (The experimental method of determining such a curve is described in Chapter IV.) From this curve it is seen that at first, when the field intensity is small, the flux density increases relatively slowly as the field intensity increases. When the field intensity is increased further, the flux density increases very rapidly; when the field intensity becomes still greater, the flux density increases more and more slowly, and finally any further increase in the field intensity produces only a comparatively slight change in the flux density. When the field intensity is now reduced the flux density instead of returning to the same values it had for the increasing values of the field intensity, *decreases less rapidly than it increased*, that is,

the decreasing values of the flux density *lag behind* the values corresponding to an increasing field intensity. This phenomenon

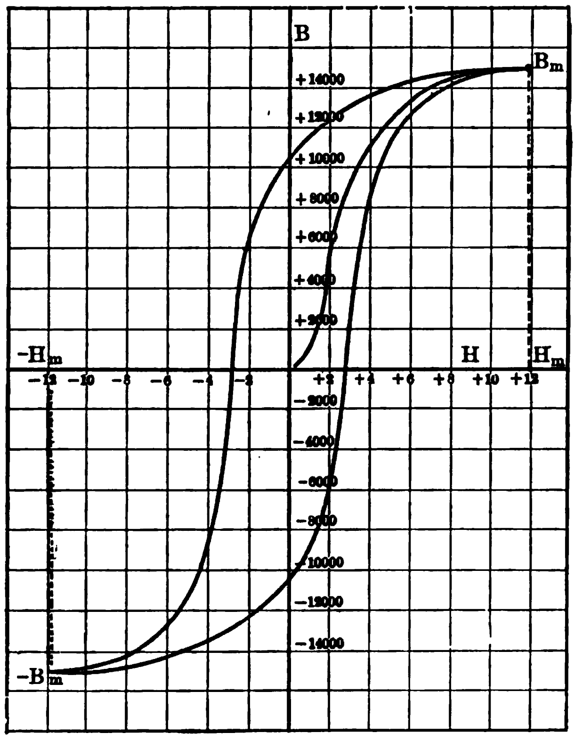


Fig. 36.

has therefore been given the name *hysteresis*, a Greek word meaning "lagging behind."

When the field intensity is reduced to zero, the flux density still has a considerable value; this value is called the "remanent magnetism" of the sample. To reduce the flux density to zero, the field intensity has to be reversed and increased to a value corresponding to the abscissa of the point where the left-hand branch of the curve cuts the axis of field intensities. This value of the field intensity is called the "coercive force." When the field intensity is still further increased in the negative direction the flux density reverses in direction, increasing very rapidly at first and then more slowly. When the field intensity reaches the same value in the negative direction as its maximum

value in the positive direction, the flux density likewise reaches a maximum in the negative direction equal to the maximum value it had in the positive direction. When the field intensity is now reduced to zero and then increased again in the positive direction to its original maximum value, the flux density passes through the series of values represented by the right-hand branch of the curve, which is perfectly symmetrical with the left-hand branch. The closed curve formed by these two branches is called the " hysteresis loop."

It should be noted that when the sample is originally unmagnetised the curve giving the relation between the flux density and the field intensity when the field intensity is first increased from zero up to the maximum value, does not coincide with either branch of the hysteresis loop, but is a curve which in general lies between the two branches of the loop. After the completion of a cycle of changes in the field intensity from a positive maximum to an equal negative maximum and then back again to the positive maximum, the field intensity may then be reversed back and forth any number of times between these equal positive and negative maximum values, and the relation between the flux density and the field intensity will be the same for each cycle of changes in the field intensity as for the first cycle. If the iron is not originally unmagnetised, the first hysteresis loop will be shifted above or below the axis of field intensities, but after a number of reversals of the field intensity between given positive and negative values, the loop will become practically symmetrical with this axis, particularly if the iron is continually jarred. In the armatures of electrical machines and the cores of transformers, in which the field intensity reverses a large number of times every second and the iron is continually jarred, the relation between flux density and field intensity, after a short interval of time, is represented by a symmetrical loop of the form shown in Fig. 36.

The area enclosed by the hysteresis loop depends upon the maximum value of the flux density reached during the cycle, but the general shape remains about the same. Fig. 37 shows a series of loops corresponding to various values of the maximum flux density. The area of the loop is also different for various kinds of iron or steel. As we shall see when we come to the study of electric currents (Chapter IV), the area of this loop represents a certain amount of energy dissipated as heat energy in the iron; in fact, the

ergs of heat energy dissipated per cycle is equal to $\frac{1}{4\pi}$ times the area of this loop, when both the flux density and the field intensity are plotted to the same scale.

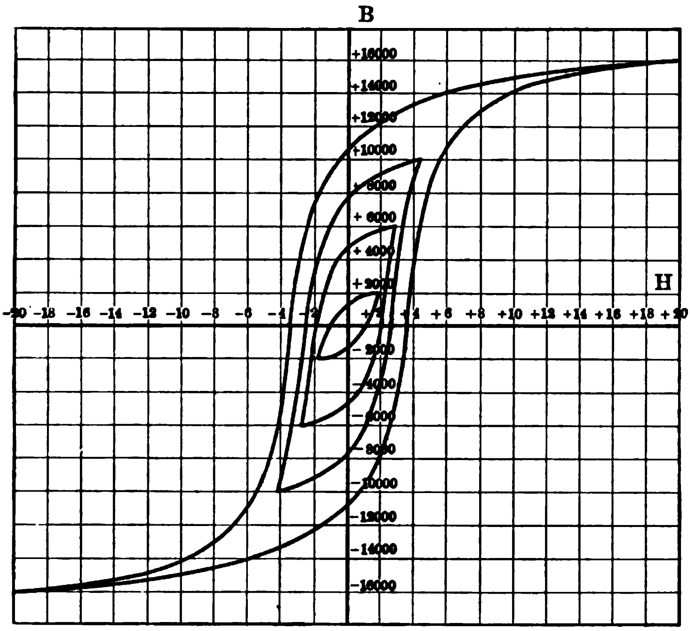


Fig. 37.

On account of this energy loss due to hysteresis, it is desirable to have all parts of the magnetic circuit of an electric machine in which there is a varying magnetic flux made of iron or steel in which this hysteresis loss is a minimum. It has been found that in iron which contains about three per cent silicon this hysteresis loss is about half what it is in the best grade of low carbon steel. This so-called " silicon steel " is now being extensively used in the construction of electric machines, particularly transformers.

An examination of this hysteresis loop also makes clear how a bar of steel may be permanently magnetised by placing it in a magnetic field. For, when the bar is removed from the field it retains an intensity of magnetisation approximately equal to the flux density where the hysteresis loop cuts the axis of flux densities, that is, an intensity of magnetisation approximately equal to the remanent magnetism. Experiment shows that a hard steel bar

thus magnetised may be handled with comparative roughness without reducing to any considerable extent the strength of its poles, but in the case of a soft iron bar even the slightest jar will cause it to lose its magnetism almost entirely. The property possessed by a hard steel bar of retaining its magnetisation is called its "retentiveness."

58. Normal B-H Curves. — Magnetic Saturation. — The curve giving the relation between the flux density and the resultant field intensity when the latter is increased from zero up to successively

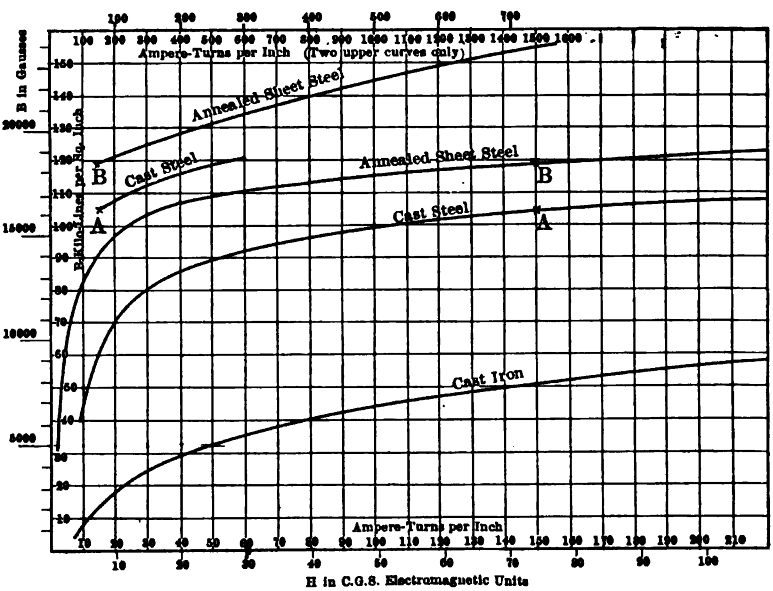


Fig. 38.*

greater values is called the "normal B-H curve." In Fig. 38 are given the normal B-H curves for cast iron, cast steel and annealed sheet steel (low carbon) such as is ordinarily used in the construction of electric machines. In Fig. 39 are given the corresponding curves showing the relation between the intensity of magnetisation and the field intensity (calculated from the B-H curves by equation 18), and in Fig. 40 the corresponding permeability curves (calculated from the B-H curves by equation 19). It will be noted that the intensity of magnetisation corresponding to values of the flux density above the sharp bend or knee in the B-H curves,

*Standard curves used by General Electric Co.

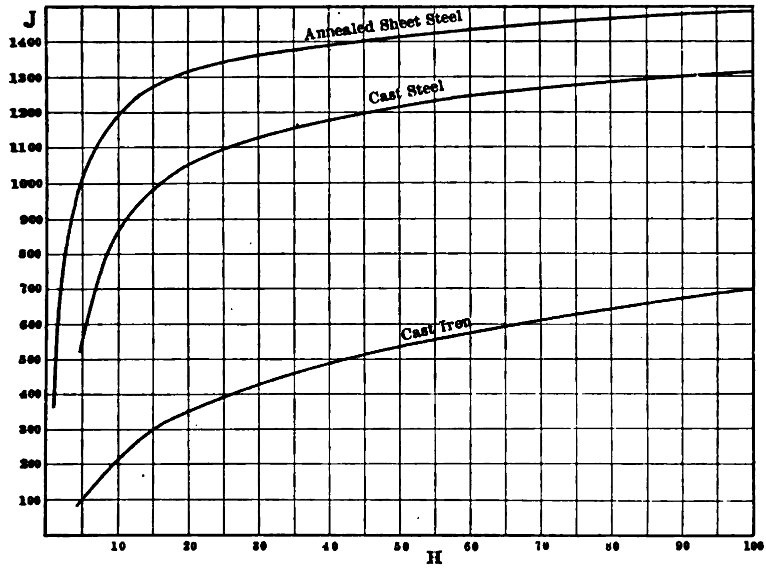

J
1400
1300
1200
1100
1000
900
800
700
600
500
400
300
200
100
Annealed Sheet Steel
Cast Steel
Cast Iron
10
20
30
40
50
60
70
80
90
100
H

Fig 39.

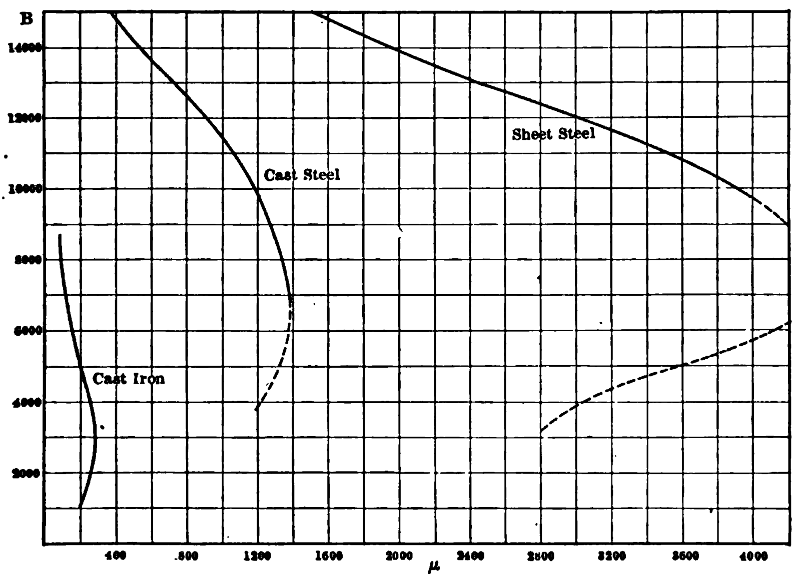

B
14000
12000
10000
8000
6000
4000
2000
Sheet Steel
Cast Steel
Cast Iron
400
800
1200
1600
2000
2400
2800
3200
3600
4000
μ

Fig. 40.

increases very slowly and soon becomes practically constant independent of the value of the field intensity. In fact, experiment shows that it is impossible to produce an intensity of magnetisation in a given substance greater than a certain definite value, which is different for different substances. When a magnetic substance is thus magnetised to its maximum intensity of magnetisation it is said to be "saturated." Such a substance is practically saturated for any value of the field intensity well above the knee of the B-H curve.

It should be noted that the B-H curves for iron and steel depend to a very great extent upon the physical structure and chemical constitution of the sample, and the heat treatment to which it has been subjected. It has also been recently discovered that when iron is annealed in an alternating magnetic field, the permeability is increased in certain cases as much as 50 per cent. The B-H curves of two samples taken from the same lot of material may even differ considerably. The curves also depend upon the temperature of the sample at the time the observations are taken, though the variation due to ordinary changes of temperature is slight. For very high temperatures, however, all magnetic substances become practically non-magnetic. This temperature corresponds to the major recalescence point, which is about 750 degrees for steel of the quality used in armature and transformer punchings. When iron is kept continuously at a moderately high temperature (100°C), the hysteresis loop also gradually increases in size, and therefore the energy loss in the magnetic circuits of electric machines due to "hysteresis" increases with time. This effect is called "aging." There is practically no aging of silicon steel.

59. Magnetic Potential. — Whenever the position of a magnetic pole with respect to another pole is changed, work must in general be done, either by the poles themselves or by the external agent which produces this change in the relative position of the two poles. For, every magnetic pole produces a force on every other pole, and consequently when one pole moves with respect to the other there is a force and a displacement, and, by definition, work is equal to the product of the displacement by the component of the force in the direction of the displacement (Article 21). The total work which two permanent poles of like sign at a given distance apart can do on each other, is the work done when the two poles

move from their original positions to an infinite distance apart. This amount of work may be looked upon as the relative potential energy of the two poles, since it is the work they are capable of, or have the "potentiality" of, doing on each other. Similarly, the work done by any number of magnetic poles in moving a unit north point-pole from a given point to an infinite distance may be looked upon as the relative potential energy with respect to these poles of a unit north pole located at this point. This quantity is called simply the *potential* at the point; *i.e.*, the potential at any point in a magnet field is the work done by the poles producing the field in moving a unit north point-pole from that point to infinity, provided these poles remain constant in strength and their relative positions remain unaltered. Potential is then work divided by pole strength, and since both work and pole strength are scalar quantities, potential is also a scalar quantity. Hence potentials may be added algebraically.

The potential at any point P due to a point-pole of strength m at a distance r away can be readily calculated. Consider any

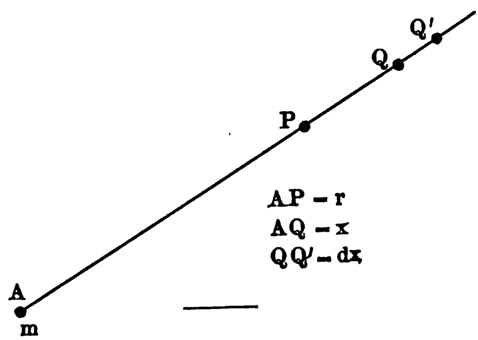


Fig. 41.

point Q on the line drawn through the pole m and the point P; let this point be at a distance x from m. Then the force which would act on a unit north pole at Q is $\frac{m}{x^2}$ in the direction from m to Q. When the unit pole moves a distance dx along this line, the work done by m on it is $\frac{mdx}{x^2}$. Consequently, when the unit pole moves along this line from P to infinity the work done is

$$\int_r^\infty \frac{mdx}{x^2} = m\left[-\frac{1}{x}\right]_r^\infty = \frac{m}{r}$$

Hence the potential at any point due to a point-pole m at a distance r away is

$$V = \frac{m}{r} \qquad (24)$$

When m is a south pole, $\frac{m}{r}$ is negative, which means that the unit pole does work *against* the force due to m, instead of this force doing work *on* the unit pole. It can readily be shown that the work done by the pole m on the unit north pole, when the latter moves from the point P to infinity, is independent of the path over which the unit pole moves. For, the work done on the unit pole when it moves a distance dl in any direction making an angle θ with the line from m to Q is $\frac{m}{x^2}(dl \cos \theta)$. But $dl \cos \theta$ is equal to the projection of dl on the line from m to Q. Calling this projection dx, we have that the work corresponding to the displacement dl is $\frac{mdx}{x^2}$, which integrated between the limits $x=r$ and $x=\infty$, gives the same value of the potential as before. Hence the potential at any point due to a point-pole of strength m depends only on the position of this point with respect to m and the value of m.

The potential at any point due to any number of point-poles m_1 m_2 etc., at the distances r_1 r_2 etc., from this point is then the algebraic sum

$$V = \frac{m_1}{r_1} + \frac{m_2}{r_2} + \text{———} \qquad (24a)$$

and the potential at any point due to any magnetised surface is

$$V = \int_S \frac{\sigma\, ds}{r} \qquad (24b)$$

where ds is any elementary area in this surface, σ the pole strength per unit area at ds, and r the distance of ds from the point considered, and $\int_S$ indicates the sum of the expressions $\frac{\sigma\, ds}{r}$ for all the elements into which the surfaces are divided.

60. Difference of Magnetic Potential. — The difference of potential between any two points 1 and 2 in a magnetic field, or

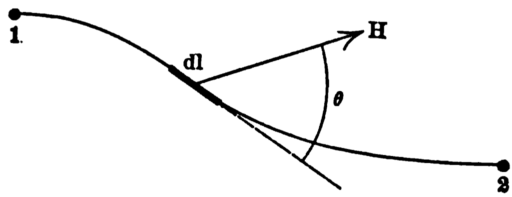


Fig. 42.

specifically, the *drop* in magnetic potential from the point 1 to the point 2, is equal to the work that would be done by the agents producing the field were a unit north point-pole moved from 1 to 2. Let V_1 and V_2 be the potentials at the two points respectively, then the drop of potential from 1 to 2 is $V_1 - V_2$. Let dl represent an elementary length in the path over which the unit pole moves, H the field intensity at dl and θ the angle between dl and the direction of the field intensity at dl; then the work that would be done by the agents producing the field on a unit north point-pole when the latter moves from 1 to 2 is $\int_1^2 (H \cos \theta)\, dl$ where

$$V_1 - V_2 = \int_1^2 (H \cos \theta)\, dl \qquad (25)$$

When V_2 is greater than V_1 the drop of potential from 1 to 2 is negative, that is the integral $\int_1^2 (H \cos \theta)\, dl$ is a negative quantity. A negative drop of potential in any direction is equivalent to a positive rise in that direction.

When the magnetic field is due solely to magnetic poles the drop of magnetic potential around any closed path is zero, since the magnetic potential at any point due to any number of poles can have but a single value. This, however, is not true when the path links an electric current (see Chapter IV).

When the two points 1 and 2 are an infinitesimal distance dl apart, the drop of potential along dl is likewise an infinitesimal quantity and may be written $-dV$ where dV stands for an infinitesimal *increase* of potential along dl and therefore $-dV$ for a *decrease* or *drop* of potential along dl. We then have that

$$-dV = (H \cos \theta)\, dl$$

or

$$H \cos \theta = -\frac{dV}{dl} \qquad (25a)$$

That is, the component in any direction of the field intensity at any point is equal to the negative of the "space rate" of change of the potential in that direction. When the elementary length is taken in the direction of the field intensity equation (25*a*) becomes

$$H = -\frac{dV}{dl_t} \tag{25b}$$

where dl_t means an elementary length taken tangent to the line of force through this point. Hence a large field intensity corresponds to a rapid fall of potential in the direction of the field intensity, and a small field intensity a gradual fall of potential in the direction of the field intensity. Consequently, the expression "potential gradient" is frequently used for field intensity, where by "potential gradient" is meant the drop of potential per unit length in the direction of the lines of force. The unit of magnetic potential difference in the *c. g. s.* system is called the *gilbert.* Hence magnetic field intensity may be expressed as so many *gilberts per centimeter.*

When the magnetic field is produced by an electric current it is usual to express the drop of magnetic potential as so many "ampere-turns" (see Chapter IV.). The relation between gilberts and ampere-turns is

1 gilbert = 0.79578 ampere-turns.

61. Equipotential Surfaces. — A magnetic equipotential surface is a surface drawn in a magnetic field in such a manner that the drop of magnetic potential along any path in the surface is zero. Such a surface is perpendicular at each point to the line of force through that point, otherwise the field intensity at the point in question would have a component along the surface and consequently there would be a difference of potential between this point and the neighboring point in the direction of this component. The lines of force representing a magnetic field are therefore always normal to any equipotential surface which may be drawn in the field. Calling dn an elementary length measured outward along the *normal* to such a surface at any point, the field intensity at this point is

$$H = -\frac{dV}{dn} \tag{26}$$

which is the same as equation (25*b*), only expressed in a different form.

SUMMARY OF IMPORTANT DEFINITIONS AND PRINCIPLES

1. A **unit point-pole** is a pole which repels with a force of one dyne an equal point-pole placed one centimeter away.

2. Two point-poles of strengths m and m' at a distance r centimeters apart repel each other with a force of

$$f=\frac{m\,m'}{r^2}\text{ dynes.}$$

3. The **field intensity** at any point of a magnetic field is defined as the force in dynes which would act on a unit north point-pole placed at that point **due solely** to the agents (magnetic poles or electric currents) producing the **original** field. The unit of field intensity in the *c. g. s.* system is the **gilbert per centimeter.**

4. The field intensity at a distance r from a point-pole of strength m is

$$H=\frac{m}{r^2}\text{ gilberts per cm.}$$

5. The mechanical force exerted on a point-pole of strength m is

$$F=mH\text{ dynes}$$

where H is the field intensity in gilberts per cm. at the point occupied by m due to all the poles (and electric currents) in the field **except the pole m.**

6. The normal component of the field intensity at any point P just outside a magnetically charged surface due solely to the pole on this surface is

$$H=2\,\pi\,\sigma\text{ gilberts per cm.}$$

where σ is the pole strength per unit area at the point on the surface directly opposite P.

7. The **magnetic moment** of a magnet is defined as the ratio of the maximum moment exerted on a magnet, when placed in a **uniform** magnetic field, to the intensity of this field. The magnetic moment of a bar magnet of length l with poles of strength m and $-m$ concentrated in points at its two ends is ml.

8. The frequency of vibration of a magnetic needle suspended in a magnetic field is proportional to the square root of the field intensity.

9. **Lines of magnetic force** are lines drawn in a magnetic field in such a manner that they coincide in direction at each point P with the field intensity at P and their number per unit area at each

point P across a surface at right angles to their direction at P is equal to the field intensity at P.

10. The number of lines of force, or flux of force, crossing any elementary area ds is

$$d\psi = (H \cos \alpha)\, ds$$

where H is the field intensity at ds and α is the angle between the direction of H and the normal to ds; or $H \cos \alpha$ is the component of H normal to ds.

11. **Gauss's Theorem**:— The algebraic sum of the number of lines of force outward from any closed surface is equal to 4π times the algebraic sum of the poles inside this surface, *i. e.*,

$$\int_{|S|} (H \cos \alpha)\, ds = 4\pi \Sigma m$$

where $\int_{|S|}$ represents the surface integral over the closed surface S.

12. A magnetised body is considered to be made up of **magnetic filaments** such that were the lateral walls of any one of these filaments separated from the rest of the body by a narrow air gap, no poles would be formed on these lateral walls.

13. The **intensity of magnetisation** J at any point of a magnetised body is the value of the strength per unit area of the pole which would be formed on the walls of a gap cut in the body at this point perpendicular to the direction of the magnetic filament through this point. The direction of the intensity of magnetisation is the direction of the filament, and the positive sense of the filament is the sense of the line drawn **into** the gap **from** the wall on which the **north** pole is formed.

14. A **line of magnetisation,** or unit magnetic filament, is a filament of such a size that were it broken by a narrow gap at any point, the strength of the pole formed on either wall of the gap would be $\frac{1}{4\pi}$.

15. The number of lines of magnetisation crossing any elementary area ds is

$$dN = 4\pi (J \cos \alpha)\, ds$$

where J is the intensity of magnetisation at ds and α is the angle between the direction of J and the normal to ds.

16. The number of lines of **magnetic induction,** or flux of induction, crossing any surface is defined as the algebraic sum of

the lines of magnetisation and the lines of force crossing that surface. The direction of the line of induction at any point is the direction of the vector which is equal to the vector sum of $4\pi J$ and H at this point. The unit of flux of induction in the *c. g. s.* system is the **maxwell**; one line of induction is equal to one maxwell.

17. **The flux density** B at any point is defined as the number of lines of induction per unit area crossing an elementary surface drawn at this point normal to their direction. The flux density is the vector sum

$$B=\overline{4\pi J+H.}$$

The unit of flux density in the *c. g. s.* system is the **gauss**.

18. The number of lines of induction crossing any elementary area ds is

$$d\phi=(B\cos a)\,ds$$

where a is the angle between the direction of B and the normal to ds.

19. A line of force originates at a north pole and ends at a south pole and may exist in either a magnetic or non-magnetic substance. A line of magnetisation originates at a south pole and ends at a north pole and exists in a magnetic substance **only**. A line of induction is a closed loop without ends and may exist in either a magnetic or non-magnetic substance. In a non-magnetic substance lines of force and lines of induction are identical, in a magnetic substance they are never identical.

20. The **magnetic permeability** μ of a body is the ratio of the flux density B established in the body to the resultant field intensity H, when the body, originally unmagnetised, is placed in a magnetic field which is continually increased from zero to the value H; that is

$$\mu=\frac{B}{H}$$

The flux density is not a constant but depends upon the value of H.

21. When the field intensity in a magnetic body is changed from one value to another and then back again to its original value, the flux density does not return to its original value. This phenomenon is known as **magnetic hysteresis.** As a result of this phenomenon a certain amount of heat energy is always dissipated in a magnetic body when it is subjected to a varying magnetic field.

22. The **magnetic potential** V at any point in a magnetic field is the work which would be done by the agents producing the field in moving a unit north point-pole from this point to infinity. The unit of magnetic potential in the *c. g. s.* system is the **gilbert**. The potential at any point due to a point-pole of strength m at a distance r centimeters away is

$$V = \frac{m}{r} \text{ gilberts.}$$

The resultant potential due to any number of poles is the algebraic sum of the potentials due to all the individual poles.

23. The **drop of potential** from any point 1 to any point 2 is

$$V_1 - V_2 = \int_1^2 (H \cos \theta)\, dl \text{ gilberts}$$

where dl is the length in centimeters of any element of the path from 1 to 2, H the field intensity in gilberts per cm. at dl, and θ the angle between the direction of H and dl. When the field is due solely to magnetic poles the drop of potential is independent of the path from 1 to 2. This is not true when the path links an electric current.

24. A **magnetic equipotential surface** is a surface drawn in a magnetic field in such manner that the drop of magnetic potential along any path in the surface is zero. Such a surface is perpendicular at each point to the line of force through that point.

PROBLEMS

(Note: In calculating forces and field intensities when a slender bar magnet is specified, the poles are to be considered concentrated in points at its ends.)

1. Two slender bar magnets each with poles of 100 *c. g. s.* units and 10 inches in length lie upon the same straight line with their centers 15 inches apart. What is the force in dynes exerted by one magnet on the other if the nearest poles on the respective magnets are of opposite signs?

Ans.: 50.7 dynes.

2. Two slender bar magnets A and B are placed parallel to each other and 10 inches apart with their centers on a line perpendicular to their axes and with opposite poles on the respective magnets adjacent. The poles of A are 50 *c. g. s.* units and the poles of B are 30 *c. g. s.* units. A is 20 inches in length and B is 10

inches in length. What is the amount (in dynes) and the direction of the total force exerted by one magnet on the other?

Ans.: 2.52 dynes in the direction perpendicular to the axis of each magnet.

3. Find the amount and direction of the field intensity due to a slender bar magnet at a point 10 cm. from the magnet on a line normal to the axis of the magnet through its north pole. The magnet is 10 cm. in length and the strength of each pole is 50 *c. g. s.* units.

Ans.: 0.368 gilberts per cm. at an angle of 61.3° with the axis of the magnet.

4. A slender bar magnet with poles of 25 *c. g. s.* units is placed in a uniform field the intensity of which is 100 gilberts per cm. What is the force in dynes acting upon each pole? What is the total force acting on the magnet?

Ans.: 2500 dynes on each pole; total force zero.

5. Two similar bar magnets A and B are placed end on with their two nearest pole-faces $\frac{1}{32}$ inch apart and of opposite sign. Each magnet is 20 inches long and the pole strength of each pole is 100 *c. g. s.* units uniformly distributed over the respective end surfaces of the magnet. If the cross section of each magnet is 1 square inch, find the force in dynes exerted on each other by the two adjacent poles. What is the force produced by one magnet on the other due to the action of the other poles?

Ans.: An attraction of 9736 dynes. A repulsion of 6.79 dynes. (Note that this latter force is less than 0.1% of the force due to the adjacent poles.)

6. A slender bar magnet 30 cm. in length and with poles of 80 *c. g. s.* units is pivoted at its center and placed in a uniform magnetic field, the intensity of which is 300 gilberts per cm. What is the torque acting upon the magnet about its center, when the axis of the magnet is perpendicular to the field intensity? What is the magnetic moment of the magnet?

Ans.: 720,000 cm-dynes. 2400 *c. g. s.* units.

7. A slender bar magnet l centimeters long produces a field intensity of H gilberts per cm. at a point on a line through the axis of the magnet at a distance of $10l$ centimeters from its center. What is the field intensity produced by this magnet at a point on a line through the center of the magnet perpendicular to its axis and at the same distance l from its center?

In solving this problem note that $\left(\frac{1}{20}\right)^2$ is negligible compared with unity.

Ans.: $\frac{H}{2}$ gilberts per cm. Note: The ratio of these two intensities depends solely upon the exponent of r in the fundamental formula, equation (1), for the mutual action of two poles. Gauss determined this ratio experimentally to be 2:1, which was the first accurate experimental proof of the inverse square law.

8. A slender bar magnet 5 square centimeters in cross section and 30 centimeters in length has a pole strength of 1500 *c. g. s.* units. Find the field intensity and the flux density at the center of the magnet; state the direction of each.

Ans.: 13.3 gilberts per cm. in the direction from north to south pole of the magnet. 3757 gausses in the direction from south to north pole of the magnet.

9. A slender iron bar 3 sq. cm. in cross section and 40 cm. in length is placed in a uniform magnetic field, the intensity of which before the introduction of the bar is 160 ampere-turns per inch. After the bar of iron is placed in this field with its axis parallel to the direction of the field intensity, a uniform intensity of magnetisation of 1000 *c. g. s.* units is established in the bar (*i.e.*, the lines of magnetisation are to be assumed straight lines parallel to the axis of the bar). Calculate the field intensity in gilberts per cm. and the total flux of induction at the center of the bar; state the direction of each.

Ans.: 64.2 gilberts per cm. in the direction from the south to the north pole of the bar. 37,773 maxwells in the same direction as field intensity.

10. The field intensity at the center of a slender bar magnet 100 cm. long and 4 sq. cm. in cross section is 500 gilberts per cm. If the flux density at the center of the magnet is 5000 gausses and is parallel to the direction of the field intensity, what is the strength of each pole, assuming the lines of magnetisation to be straight lines? What would be the field intensity in the region occupied by the magnet if the magnet were removed?

Ans.: 1432 *c. g. s.* units. 501.15 gilberts per cm.

11. A slender magnet has a magnetic moment of 5000 *c. g. s.* units. The magnet is 50 cm. in length and 2 sq. cm. in cross

section. Calculate the intensity of magnetisation and the flux of magnetisation at the center of the rod.

Ans.: 50 *c. g. s.* units. 1257 *c. g. s.* units.

12. A slender bar of iron 50 cm. long and 5 sq. cm. in cross section is placed in a uniform field the intensity of which is 50 gilberts per cm., the axis of the iron bar being parallel to the direction of the field intensity. If the permeability of the iron at the degree of saturation attained is 300, find the intensity of magnetisation and the flux density at the center of the bar.

Ans.: 862 *c. g. s.* units. 10,870 gausses.

13. Fig. 36, p. 83, represents the hysteresis loop for a sample of iron. Calculate the torque required to revolve a sample of this iron having a volume of 50 cu. cm. at a uniform speed of 300 revolutions per minute in a uniform magnetic field which has an intensity of 12 gilberts per cm. The demagnetising action of the poles induced on the iron and mechanical friction are to be neglected.

To solve this problem equate the energy dissipated per second in the iron due to hysteresis to the mechanical power expressed in terms of torque and angular speed.

Ans.: 1770 cm-dynes. (Note that this torque is independent of the speed, but depends only upon the area of the hysteresis loop, *i.e.*, the torque is $T = \frac{A}{8\pi^2}$. Hence the torque is proportional to the hysteresis loss per cycle of variation of field intensity. This is the principle of Ewing's hysteresis tester.)

14. Three equal north point-poles of 25 *c. g. s.* units each are placed at the vertices of an equilateral triangle, each side of which is 5 inches in length. Find (1) the intensity of the field at the center of the triangle; (2) the magnetic potential at the center of the triangle; (3) the intensity of the field at a point on a line through any two poles midway between them; and (4) the magnetic potential at a point on a line through any two poles midway between them. South poles may be neglected.

Ans.: (1) 0. (2) 26.0 gilberts. (3) 1.33 gilberts per cm. in the direction away from the third pole. (4) 25.8 gilberts.

15. The numerical strength of each pole of a round bar magnet is 200 *c. g. s.* units, and these poles are uniformly distributed over its end surfaces. The cross section of the magnet is 1 sq. in. and its length is 10 inches. Calculate the magnetic potential at the

center of the north pole of the magnet. In making this calculation find first the potential at this point due to the north pole considered as a uniformly magnetically charged disc and then add (algebraically) to this the potential due to the south pole considered as a point-pole. Why is this approximation justified? What is the drop of magnetic potential from one pole to the other? Is the drop through the magnet the same as the drop between these two points through the air around the magnet?

Ans.: 1249 gilberts. Since the angle subtended by the south pole is small, see Article 37. 2498 gilberts. Yes.

III

CONTINUOUS ELECTRIC CURRENTS

62. The Electric Current. — When a strip of zinc is dipped into a dilute solution of sulphuric acid in water, hydrogen gas is given off from the strip, but if the zinc is pure, this action ceases almost immediately. A copper strip dipped into the same solution is not appreciably affected, provided the two strips

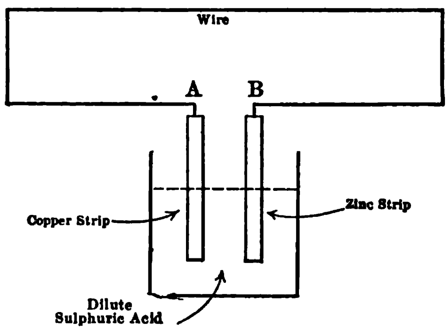


Fig. 43.

do not touch each other. Now connect the ends of the two strips by a copper wire (Fig. 43); the following phenomena are then observed:

1. The zinc gradually wastes away and hydrogen gas is liberated at *the copper strip.*

2. The wire, the strips, and the solution all become heated.

3. A magnetic field is produced around the wire, the strips and the solution.

4. A magnet placed in the vicinity of the apparatus will exert a mechanical force on the wire, the strips, and the solution.

5. When the copper wire connecting the two strips is broken in two and the two ends *a* and *b* (Fig. 44) are both dipped into a solution of copper sulphate, it is found that the wire *a* wastes away where it is immersed in the copper sulphate solution and copper is deposited on the wire *b* where it is immersed in the solu-

tion. In addition, this solution becomes heated, has a magnetic field produced around it, and a magnet placed near it exerts a mechanical force upon it.

6. When the ends *A* and *B* of the copper wire are reversed, so

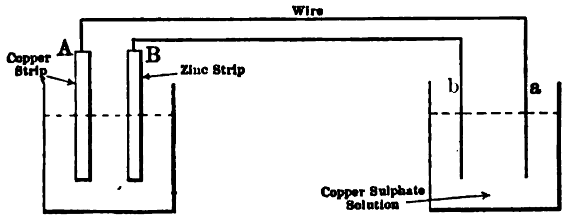


Fig. 44.

that *A* is connected to the zinc strip and *B* is connected to the copper strip, all the phenomena remain the same as before *except* that

a. The direction of the magnetic field around the wire and the copper sulphate solution is reversed.

b. The direction of the mechanical force produced on the wire and the copper sulphate solution by a magnet placed in their vicinity is also reversed.

c. The end of the wire *a* in the copper sulphate solution now has copper deposited upon it, while the end of the wire *b* in the copper sulphate solution wastes away.

These various phenomena are said to be due to a "*current of electricity*," or briefly, to an "*electric current*" in the wire, the strips and the solution. The various parts of the apparatus taken together are said to form a *closed electric circuit*, and any one of the parts is spoken of as part of an electric circuit. The combination of the copper and zinc strips and the dilute sulphuric acid solution is called an *electric battery*, and the copper and zinc strips are called the *poles* of the battery. Experiment shows that when the phenomena just described occur in and around the wire, the battery loses chemical energy, consequently the battery is to be looked upon as the cause of these phenomena, and since the phenomena are attributed to a flow of "electricity" through the various parts of the apparatus, the battery is said to produce an "electricity moving force" or, more briefly, an *electromotive force*. There are other devices which are capable of causing these phenomena to occur in and around a wire connected to

them, and any such device is said to be the seat of an electromotive force. The electric dynamo, the operation of which will be described later, is a device which is capable of producing very powerful electromotive forces and electric currents thousands of times greater than the maximum current which can be produced by a simple battery of the form just described. But before we can speak accurately of the quantitative values of electric currents and electromotive forces, it is necessary to define what shall be taken as the measure of these quantities.

63. Conductors and Insulators or Dielectrics. — First, however, it is important to note that the degree to which the effects described above are produced when different kinds of substances are used in place of the copper wire to connect the poles of the battery, depends to a very great extent upon the nature of these substances. For example, when there is nothing but air between the poles of the battery none of the effects described above are observed. Again, a dry silk string may be used to connect the poles of the battery, and no effects will be observed. When a moist string is used, the effects produced are similar to those produced when the poles are connected by the copper wire, but to a markedly less degree. Any substance which, when connected to the poles of an electric battery, has a magnetic field produced around it as long as it remains in contact with these poles, is called a *conductor* of electricity; if no magnetic field is established around the substance, it is called an *insulator* or *dielectric*. It should be noted, however, that when a battery of sufficiently high electromotive force is used, any substance connected to its poles will have a magnetic field produced around it, though to a far less extent than would be produced around a metallic wire connected to its poles. In fact, there is no known substance which is a perfect insulator, though the results of all known experiments lead us to believe that a perfect vacuum, if obtainable, would be such. However, for most practical purposes, such substances as glass, glazed porcelain, rubber, ebonite, gutta-percha, paraffine, silk, cellulose and shellac may be considered as insulators, while all metals, carbon, fused salts and solutions of most mineral salts and acids are conductors.

A conductor completely surrounded by insulators is said to be completely *insulated*. A wire is also said to be insulated when its lateral walls only are surrounded by an insulator. That is, a

rubber-covered wire, for example, with its ends connected to the poles of a battery, is spoken of as an "insulated wire connected to the battery."

64. Electricity Analogous to an Incompressible Fluid Filling all Space. — The following analogies will be found helpful in understanding the significance of the various properties which experiment leads us to assign to the something called electricity. In the first place, experiment shows that this something must have many properties analogous to those which would be possessed by an incompressible fluid filling all space, including the space occupied by matter. In this analogy, the force of gravitation acting on the fluid is to be neglected. The properties possessed by free space or any space occupied by a dielectric are found to be analogous to the properties which would be possessed by this space were the incompressible fluid in this space enclosed in minute cells with elastic walls, forming a cellular or honey-comb structure with continuous walls completely filling this space; while the properties possessed by a conductor are analogous to the properties possessed by a space in which the walls of the cells are completely destroyed. Any particle of this fluid in the space corresponding to a dielectric can then move only as the result of a strain produced in the elastic structure which enmeshes it, while any particle of the fluid in the space corresponding to a conductor can move freely throughout this space. A conductor may then be looked upon as analogous to an elastic sack (or tube, in the case of a wire) completely filled with this incompressible fluid, the elastic walls of the sack being formed by the walls of the cellular structure representing the dielectric surrounding the conductor.

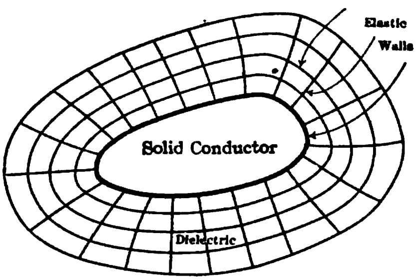


Fig. 45.

The property possessed by a battery of being capable of producing a "flow of electricity" is analogous to the pressure produced by a pump. The analogy to a simple battery of the form described above is a pump which maintains a constant pressure (approximately) in the same direction whether or not there is a flow of the fluid through the pump. Since a battery is always

made of conductors, the walls of this pump must also be considered as elastic.

A conductor connected to the two poles of a battery is then analogous to the rubber hose completely filled with water (which may be taken as an approximately incompressible fluid) with its two ends connected respectively to the outlet and intake of the pump, which is also completely filled with water. The pump will then force a current of water through the hose, and the strength

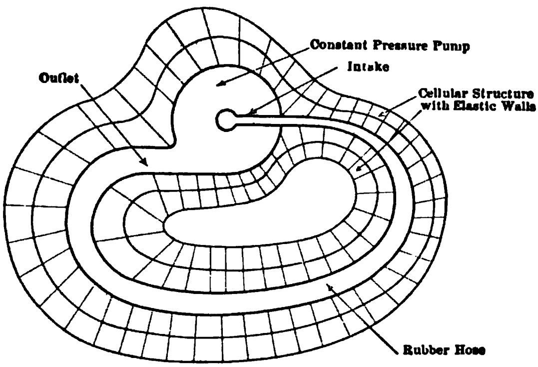


Fig. 46.

of this water current (*i.e.*, the quantity of water per unit time flowing through any cross section of the hose) will depend upon the pressure developed by the pump and the resistance due to the friction of the water against the walls of the hose. The effects produced in and around the wire connected to the battery are similarly found to depend upon a property of the battery (called its electromotive force) analogous to the pressure developed by the pump, and upon a property of the wire (called its resistance) analogous in certain respects to the resistance of the rubber hose to a water current.

When the current of water is established in the hose, a fall of pressure is produced in the hose in the direction of the flow of the water, and consequently the pressure acting on the walls of the hose will vary from point to point. Hence, near the outlet of the pump the walls of the hose will expand and near the intake the walls will contract. Consequently, there will be more water in the portion of the hose near the outlet of the pump and less near the intake, than there was in these portions of the hose before it was connected to the pump. Experiment shows that an analogous

phenomenon occurs when a wire is connected to an electric battery. The portions of the wire near the two poles of the battery manifest properties which they did not possess before; these properties are such that they may be attributed to an excess of electricity, or a "positive charge" of electricity on the portion of the wire nearer the pole of the battery *from* which the electric current is said to flow, and a deficit of electricity, or a "negative charge" of electricity, on the portion of the conductor nearer the pole of the battery *toward* which the current is said to flow.

On account of the inertia of a mass of water, time is required to set the water in the hose in motion when the latter is connected to the pump. It is also found that time is required for the effects which we ascribe to the flow of an electric current to reach a steady state. As we shall see later, the property of electricity which is analogous to inertia may be expressed in terms of the magnetic field produced by electricity in motion. There is no experimental evidence to lead us to assign to electricity a property analogous to weight.

The expansion of the walls of the rubber hose by the pressure produced by the pump likewise produces a pressure on the cellular structure representing the dielectric surrounding the wire, and produces a displacement of the cells and the fluid (*e. g.*, water) which they contain. When the current of water in the hose becomes steady, the motion of these cells ceases, but while the current in the hose is being established, there is also a motion of these cells and the water they contain. As we shall see later, an effect is produced in the dielectric while the current in the wire is being established or changes in any way, which may be attributed to a flow or displacement of electricity through the dielectric, analogous to the displacement of the water in the cellular structure surrounding the rubber hose, which displacement ceases when the electric current reaches a constant value.

The displacement of the cells of the structure surrounding the rubber hose produces a change in the shape of these cells, and therefore a strain in their walls. Analogous effects are observed in the dielectric surrounding a wire connected to a battery or other device which is capable of producing the phenomena which we attribute to the flow of an electric current. This effect is particularly noticeable when the "electromotive force" of the current-producing device is high, and may even cause a rupture of

the dielectric, *i.e.*, an electric spark. A similar effect would be produced should the pressure developed by the pump produce a strain in the walls of the cells in the structure around the hose sufficient to break the walls of these cells.

When the outlet and intake of the pump which we have been considering are closed, we have an arrangement analogous to an electric battery with its poles insulated from each other. Since our pump is assumed to maintain a constant pressure independent of the current flowing through it, and since the walls of the pump are elastic, near the outlet these walls will be stretched and near the intake contracted. Consequently there will be an excess of water near the outlet and a deficit near the intake, compared to the quantity of water that would be in these portions were there no pressure produced by the pump. The two poles of a battery possess analogous properties, which may be attributed to an excess of electricity, or "positive charge" of electricity, on the pole of the battery *from* which the current is said to flow, and a deficit of electricity, or a "negative charge" of electricity, on the pole *toward* which the current is said to flow. The cellular structure surrounding the pump will likewise be in a state of strain, due to the pressure produced in it by the pump. Experiment shows that analogous effects are also produced in the dielectric surrounding the battery.

Again, if a closed sack which has elastic walls is connected to the outlet of the pump, more water will be forced into the sack by the pump; while if this sack is connected to the intake of the pump water will be drawn out of it. These effects are analogous to the effects produced when a conductor is connected to one of the poles of a battery but not to the other. When the conductor is connected to one pole it manifests a new property which may be attributed to an increase of electricity on it, or to a "positive charge" of electricity gained by it, and when connected to the other pole it manifests a new property which may be attributed to a drain of electricity from it, or to a "negative charge" of electricity on it.

It should be noted that the above discussion is merely a statement of *analogies* and does not *explain* anything. These analogies are useful as they enable one to form a picture of the way the observed effects *might* take place, but the exact mechanism of these effects may be entirely different. Like all other analogies, the above must not be pressed too far. For example, the elastic hose

or sack which we have considered as representing a conductor, changes in shape when connected to the pump, but there is no evidence that a wire or other conductor changes in shape when it manifests the properties which are attributed to a "charge of electricity" on it.

65. A Wire as a Geometrical Line. — In the discussion of the phenomena which are attributed to the flow of an electric current a wire will usually be considered as equivalent to a geometrical line. This, of course, is not strictly accurate, since a wire always has a finite cross section. However, in many cases the error involved is practically inappreciable; when this is not so, attention will be called to the fact. The exact expression for a wire of finite cross section may always be derived when we have deduced the relation which holds for a geometrical line, for we may express this same relation for the wire of finite cross section by considering the wire made up of an infinite number of filaments of infinitesimal cross section, each of which filaments is equivalent to a line, and then determine the resultant effect due to all these filaments. In general, such an expression is extremely difficult to evaluate; only in one or two simple cases will it be necessary to do this.

66. Definition of the Strength of an Electric Current.— Definition of a Continuous Current. — The quantity of electricity that flows through any section of a wire in unit time may be called the strength of the current of electricity in this section of the wire, just as the quantity of water flowing through any section of a pipe in unit time may be called the strength of the water current in this section of the pipe. By "quantity" of water flowing through any section of a pipe is meant the *volume* of water flowing through this section; therefore quantity of water has a perfectly definite meaning and can be readily determined, either directly or by measuring its mass. There is no experimental evidence, however, to lead us to attribute to electricity either mass or volume, in the ordinary sense of these terms. To define the strength of an electric current in this manner, therefore, it would be necessary first to define what is to be meant by "quantity" of electricity. It is more convenient, however, to define the strength of an electric current in some other way, and then to define "quantity" of electricity in terms of the electric current, particularly as this method of procedure is in

accord with the usual experimental methods employed in engineering work in the determination of the "quantity" of electricity.

We might take any one of the effects described above as the measure of the strength of the current flowing in the wire. Scientists have agreed, however, to take as the measure of the strength of an electric current flowing in a wire, the mechanical force which is exerted on the wire when it is placed in a magnetic field. (This effect was illustrated above by the mechanical force produced on the wire by a magnet placed near it.) When a constant magnetic field is established at the wire by some external agent (for example, by means of a permanent magnet) it is found that in general the force exerted on the wire when it is connected to a battery of the kind described in Article 62 remains constant (at least appreciably so for several seconds or more, though, unless special precautions are taken, this force will gradually change). The current in the wire is said to be *continuous** as long as this force remains constant. This force is found to depend upon the flux density at the wire of the magnetic field produced by this external agent and also upon the direction of this flux density with respect to the direction of the wire, and upon the length of the wire. For a continuous current in the wire, the force dF produced on any elementary length dl of the wire (see Fig. 47) by a magnetic field when the flux density at dl is B and makes an angle θ with the direction of dl, is found to be proportional to the product of the flux density B, the sine of the angle θ, and the length dl; that is, the mechanical force dF is proportional to $(B \sin \theta)\, dl$. The *direction* of this force is found by experiment to be *perpendicular to the plane determined by the direction of the flux density* B *and the direction of the length* dl.

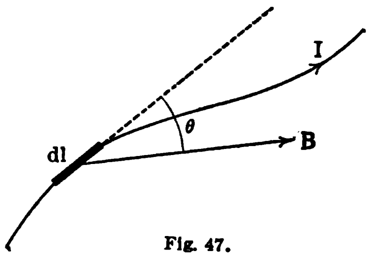


Fig. 47.

Various modifications may be made in the rest of the circuit which will cause the force acting on a given length of wire to change, even though the flux density B and the angle θ at each point is kept unaltered. For example, another piece of wire may be inserted between one end of the original wire and the pole of

*The following definitions are those adopted by the American Institute of Electrical Engineers.

the battery to which it was originally connected, that is, placed "in series" with the original wire; or a second battery may be connected in the circuit. Any modification which causes a change in the force acting on a given length of wire when the flux density B and the angle θ at each point is kept constant, is attributed to a change in the strength of the electric current in the wire. In short, the *strength* of the current in any section of the wire is arbitrarily defined to be proportional to this force when the flux density B and the angle θ at each element of this section is kept constant. We then have that the force on any elementary length dl, besides being proportional to $(B \sin \theta)\, dl$ is also proportional to the strength of the current; that is

$$dF = kI\,(B \sin \theta)\, dl$$

where I is the strength of the current in the elementary length dl of the wire, B is the flux density of the magnetic field at dl produced by any external agent, θ the angle between the direction of this flux density and the direction of dl, and k is a factor of proportionality. Experiment shows that this quantity k is independent of the nature of the substances which form the wire and the medium surrounding the wire, but depends only upon the units in which the force, the flux density, and the length are measured. The unit of current strength may therefore be chosen such that when the force dF is measured in dynes, the flux density B in gausses, and the length dl in centimeters, this factor of proportionality k is equal to unity. The above equation then becomes*

$$dF = I\,(B \sin \theta)\, dl$$

from which

$$I = \frac{dF}{(B \sin \theta)\, dl} \tag{1}$$

Note that $\dfrac{dF}{dl}$ is the mechanical force per unit length of wire at dl, and that $B \sin \theta$ is the component of the flux density at dl perpendicular to the wire at this point. Hence, as the measure of the strength of the electric current in a wire is taken the *ratio of the force per unit length of the wire which would be produced by a magnetic field to the component of the flux density of this field perpendicular to the wire.* This definition and its mathematical

*This relation is known as Biot and Savart's Law.

expression, equation (1), applies to a variable as well as a continuous current.

When the force is expressed in dynes, the length in centimeters, and the flux density in gausses, the unit of electric current strength as thus defined is called the *c. g. s. electromagnetic unit* of current, or the *absolute unit* of current, or the *abampere.* One abampere is then equal to one dyne per centimeter per gauss. In practice, a unit of one-tenth the size of this unit is employed; this practical unit is called the *ampere.* Hence

$$1 \text{ abampere} = 10 \text{ amperes.}$$

Instead of employing the expression " An electric current has a strength of so many amperes or abamperes " one usually says a current *is* so many amperes or abamperes.

When the element dl is in a non-magnetic medium, which is practically always the case in any current-measuring instrument, the flux density at dl is equal to the field intensity at this element, and consequently in this case equation (1) becomes

$$I = \frac{dF}{(H \sin \theta)\, dl} \tag{1a}$$

where H is the field intensity in gilberts per centimeter at the element dl.

67. Definition of the Direction of an Electric Current. — Left-Hand Rule. — As noted in Article 62, when the ends of the wire connected to the poles of the battery are interchanged, the force produced by any external magnetic field on the wire also reverses. An electric current must therefore be looked upon as having direction as well as magnitude. We also saw in the last paragraph that the direction of the mechanical force on each elementary length of the wire, *i.e.*, the direction in which this elementary length tends to move, is perpendicular to the plane determined by this elementary length and the direction of the flux density of the magnetic field at this elementary length. As the direction of the electric current (I) *is taken arbitrarily* the direction in which the middle finger of the left hand points when the thumb, forefinger and middle finger of this hand are held mutually perpendicular, and the thumb is pointed in the direction in which the wire tends to move and the forefinger is pointed in the direction of the component of the flux density perpendicular to the wire. This rule is called the *left-hand rule;* it is readily remembered

by noting the corresponding letters in middle and I, thumb and move, forefinger and flux.

68. Conductors in Series and in Parallel. — Experiment shows that when a number of conductors are connected end to end (Fig. 48) and are completely surrounded by insulators, then the strength of the current as above defined is the same in all these conductors, provided the current strength does not vary with

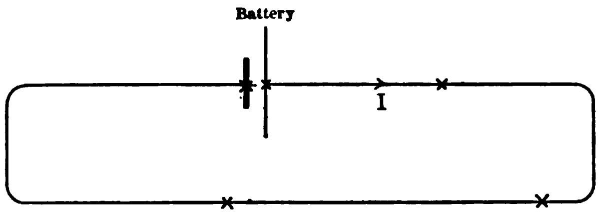


Fig. 48.

time. Two or more conductors thus connected end to end are said to be connected in *series.*

Experiment also shows that when any portion of an electric circuit between any two points A and B is formed by two or more insulated conductors (Fig. 49), the strength of the current coming up to the junction point A or leaving the junction point B is equal to the sum of the strengths of the currents in the con-

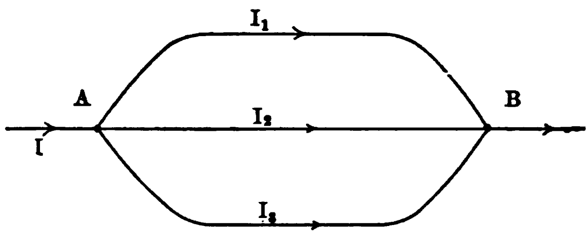


Fig. 49.

ductors joining A and B, provided the current strength does not vary with time. For example, in Fig. 49

$$I = I_1 + I_2 + I_3$$

and the conductors joining the two points A and B are said to be connected in *parallel.*

When any portion of a circuit is made up of one or more conductors connected in series with one or more groups of conductors in parallel. this portion of the circuit is said to be connected in *series-parallel.*

69. Total Force Produced by a Magnetic Field on a Wire Carrying an Electric Current. — From the above discussion and equation (1), it follows that the total mechanical force F produced by any external magnetic field on an insulated wire carrying an electric current I, when the wire is of any length l and bent into any shape whatever, is equal to the *vector* sum

$$F = I\overline{\int_0^l} (B \sin \theta)\, dl \qquad (2)$$

where dl is any elementary length of the wire measured in the direction of the current, B the flux density of the field at dl, and θ the angle between dl and the direction of B and $\overline{\int_0^l}$ indicates the *vector* sum of the expressions $(B \sin \theta)\, dl$ for all the elementary lengths into which the wire is divided. All quantities in this equation are in *c. g. s.* units. In general, the flux density and the angle θ will be different for each point of the wire. The direction of the mechanical force dF acting on each element will also be different for each element, since the plane determined by the direction of the flux density and the direction of the element will in general be different; hence the necessity for taking the *vector* sum.

70. Force Produced by a Uniform Magnetic Field on a Straight Wire Carrying an Electric Current. — One of the simplest cases is that of a straight wire in a uniform magnetic field. Consider such a wire carrying a current I, placed at an angle θ with the direction of the lines of induction; let the flux density of this magnetic field be B. In this case B and θ are constant for all points along the wire, and the mechanical force on all elements of the wire is in the same direction; the integration is then a simple algebraic one and therefore the total force on the wire is

$$F = IB \sin \theta \int_0^l dl = IB\, l \sin \theta. \qquad (2a)$$

When the wire is perpendicular to the direction of the field the force acting on the wire is

$$F = IB\, l \qquad (2b)$$

since $\theta = 90°$ and $\sin 90° = 1$. In case the wire is in a non-magnetic medium, equation (2*b*) becomes

$$F = I\, Hl \qquad (2c)$$

All quantities in these equations are in *c. g. s.* units.

71. Magnetic Field Produced by an Electric Current in a Wire. — We have seen that when a wire carrying an electric cur-

rent is placed in a magnetic field, this field exerts a mechanical force on the wire. It is also found by experiment that an equal and opposite mechanical force is exerted on the magnet or other agent producing this field; this, of course, is in accord with the general principle of nature that "action and reaction are equal and opposite." The region around a wire carrying an electric current is therefore a magnetic field of force, for by definition a magnetic field of force is any region in which a magnetic pole will be acted upon by a mechanical force. The intensity of the field of force due to a wire carrying an electric current may be readily determined from equation (1).

Consider an elementary length dl of the wire in which the current is I abamperes and let this length be measured in the direction of the current. Let the magnetic field which produces the mechanical force dF on this elementary length be that due to a unit north point-pole at any point P at a distance of r centimeters away. The flux density of the magnetic field at dl due to this unit pole is then $\frac{1}{r^2}$ *independent of the permeability of the medium surrounding the pole and the wire* (see Article 56). The angle θ between the direction of this flux density and the length dl is the angle between the direction of dl and the direction of the line drawn from the pole to dl. The mechanical force exerted by the unit point-pole on the length dl is then

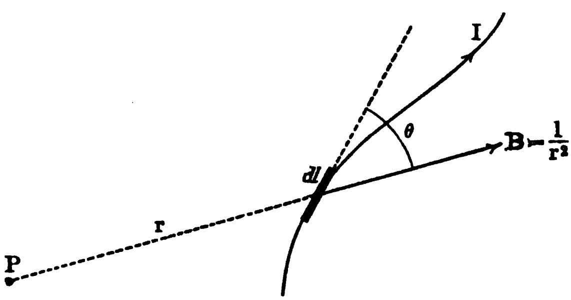


Fig. 50.

$$dF = I\left(\frac{\sin\theta}{r^2}\right)dl$$

and is perpendicular to the plane determined by the point P and the length dl. The direction of this force is determined by the left-hand rule, see Article 67; in the diagram this force is down into the plane of the paper. The current I in the elementary length dl exerts an equal and opposite force on the unit point-

pole at P. Therefore the intensity dH_i of the magnetic field at P due to the current I in the elementary length dl is

$$dH_i = \frac{(I \sin \theta)\, dl}{r^2} \tag{3}$$

independent of the permeability of the surrounding medium, and is in the opposite direction to that of the force on dl due to the pole at P. All quantities in this equation are in *c. g. s.* units. A current I flowing in an elementary length dl therefore produces a field intensity at any point P equal numerically to that produced by a point-pole of strength $(I \sin \theta)\, dl$ coinciding with dl, where θ is the angle between the direction of the current in dl and the line drawn from P to dl, *except that this intensity does not depend upon the permeability of the surrounding medium*, whereas the *resultant* intensity at P due to a point-pole at dl is inversely proportional to the permeability of the medium surrounding P and dl; see Article 56. *The direction of this intensity is also different from that of the intensity due to a pole at dl.* The latter is in the direction from dl to P, while the intensity of the magnetic field due to a current in dl is *perpendicular to the plane determined by dl and P.* Equation (3) applies to a variable current as well as to a continuous current. (In the case of a rapidly varying current, however, the value of H at any instant does not correspond to the value of I at that instant, but to the value of I at some previous instant, since time is required for the magnetic field to be propagated through space; in free space the velocity of propagation is very great, being the same as the velocity of light.)

72. Direction of the Lines of Force Produced by an Electric Current.— The lines of force due to the current I in the elementary length dl are circles which have their planes perpendicular to the straight line (OO' in the figure) drawn through dl and which have their centers on this line. For, the circumference of such a circle is perpendicular at each point to a plane drawn through this point and dl; this circumference must therefore

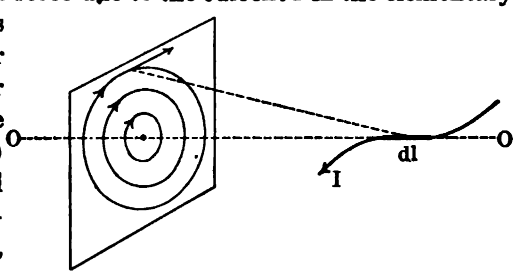


Fig. 51.

coincide in direction with the field intensity at this point; see the preceding article. From the deductions of the preceding article it also follows that the positive sense of these lines of force is the same as that in which a right-handed screw placed along the wire at dl must be turned to advance it in the positive sense of the current. When the positive senses of two quantities are thus related, the quantities are said to be in the right-handed screw direction with respect to each other.

The current in every other elementary length of the circuit will likewise produce a magnetic field, the lines of force of which are circles which have a like relation to the current in the elementary length which produces them. The lines of force representing the *resultant* field due to a current in any *finite* length of wire will not in general be circles, but it can be shown that each of the resultant lines of force is a *closed* loop which " links " the wire *in the right-handed screw direction with respect to the current.*

Note an important difference between the lines of force due to an electric current and those due to magnetic poles: *the lines of force due to magnetic poles are not closed but end on the poles, while the lines of force due to an electric current are always closed loops, linking the circuit which produces them.* When there is any magnetic body in the vicinity of an electric current this body will in general have poles induced on it by the magnetic field due to the current, and part of the lines of force representing the resultant field will end on these poles and the rest will be closed loops linking the electric circuit. The lines of induction are always closed loops whether or not there are magnetic poles in the field.

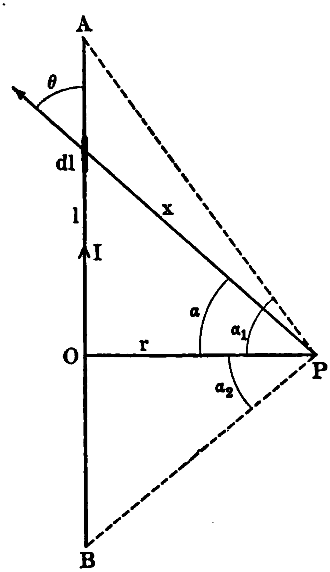


Fig. 52.

73. Magnetic Field Due to a Current in a Long Straight Wire.—A useful application of equation (3) is the calculation of the intensity of the magnetic field at any point due to a current in a long straight wire.

In Fig. 52 let BA be a straight wire and let P be any point at a distance r from it (measured perpendicular to the wire), I the current in the wire from B to A in abamperes, dl any elementary length in the direction of I at a distance l from the point O where the perpendicular from P cuts the wire, θ the angle between dl and the line from P to dl, and x the distance from P to dl. Each element of the wire will produce a field intensity at P perpendicular to the plane of the paper, downward, and therefore the field intensities at P due to all the elements of the wire may be added algebraically. The field intensity at P due to the current in the length dl is, from equation (3),

$$dH = \frac{(I \sin\theta)\, dl}{x^2}$$

and therefore the total field intensity at P due to all the elements of the wire is

$$H = \int_{-l_2}^{l_1} \frac{(I \sin\theta)\, dl}{x^2}$$

where $l_1 = OA$ and $l_2 = OB$. The simplest way to evaluate this integral is to express the different variable quantities in terms of the variable angle α (see Figure 52). From the figure we have $\sin\theta = \cos\alpha$; $x = \frac{r}{\cos\alpha}$ and $l = r\tan\alpha$. Differentiating the latter expression we get $dl = \frac{r\, d\alpha}{\cos^2\alpha}$. Let α_1 be the numerical value of the angle OPA and α_2 the numerical value of the angle OPB. Substituting these values in the above equation we get

$$H = \int_{-\alpha_2}^{\alpha_1} \frac{(I\cos\alpha)\, d\alpha}{r} = \frac{I}{r}\Big[\sin\alpha\Big]_{-\alpha_2}^{\alpha_1} = \frac{I}{r}\left(\sin\alpha_1 + \sin\alpha_2\right) \qquad (4)$$

All quantities in this equation are in $c.\ g.\ s.$ units.

When the wire is of infinite length (or practically, when the distance of the point P from the ends of the wire is great compared to the perpendicular distance r) α_1 and α_2 become equal to 90°, and therefore $\sin\alpha_1 = 1$ and $\sin\alpha_2 = 1$. We then have that the field intensity at a distance r from a straight wire of infinite length carrying a current I is

$$H = \frac{2I}{r} \qquad (4a)$$

All quantities in this equation are in *c. g. s.* units. The direction of the field intensity is perpendicular to the plane drawn through the wire and the point; the lines of force are therefore circles with their centers along the wire and their planes perpendicular to the wire. The relation between the direction of these lines of force and the direction of the current is conveniently remembered by the rule that the lines of force are in the direction in which a right-handed screw must be turned to advance it in the direction of the current. Equations (4) apply to a variable as well as to a continuous current.

The above formula has been deduced on the assumption that the wire may be considered as a geometrical line. It can also be shown that the field intensity at any point P at a distance from r from a long *circular* wire of finite cross section is also given by the above formula provided the point P lies *outside* the wire, and the current density is uniform over the section of the wire; see Article 104. At any point *inside* such a wire the field intensity is also perpendicular to the plane through the point and the axis of the wire but is equal to

$$H_i = \frac{2\,I\,r}{a^2} \tag{4b}$$

where r is the distance of the point from the center of the wire, a is the radius of the wire and I is the total current in the wire, and all quantities are in *c. g. s.* units. Experiment justifies assumption that the current density in a wire is constant, provided the wire is of uniform structure and the current is a *continuous* current. This is not true when the current varies rapidly with time, but is approximately true for ordinary variable or alternating currents used in practical work, provided the wire is non-magnetic and has a diameter less than one inch. The above formula for the field intensity outside a wire is also approximately true for a wire which has a cross section of any shape, provided the distance of the point from the wire is great compared to the greatest diameter of the wire.

74. Magnetic Field Due to an Electric Current in a Circular Coil of Wire. — Another useful application of equation (3) is the calculation of the intensity of the magnetic field at any point due to a current in a circular coil of wire. The solution of this problem except for points on the axis of the coil is quite difficult; the solution for a point on the axis of the coil is obtained as follows.

Let the coil have but a single turn and let us consider the wire forming the coil as a geometrical line making a circle of radius r. Let P be any point on the axis of this coil at a distance a from the plane of the coil. Let I be the current in the wire, dl any elementary length in the circumference of the wire and dl' an equal elementary length diametrically opposite dl. The current in dl' will be in the opposite direction from that in dl. In Fig. 53 let the current be

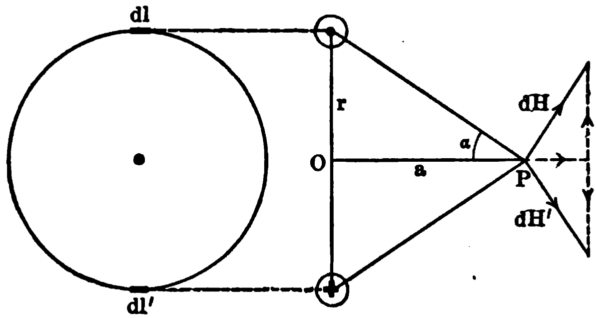


Fig. 53.

up toward the reader at dl and down at dl'. (The standard convention for showing a current coming up is a circle with a dot in it, and for a current going down a circle with a cross in it.)

The field intensity at P due to the current in dl is then, from equation (3),

$$dH = \frac{I\,dl}{(a^2+r^2)}$$

since the line drawn from P to dl is perpendicular to dl, whence in equation (3) $\sin\theta = 1$. This field intensity is perpendicular to the plane through dl and P, and is therefore in the plane of the paper in Fig. 53, perpendicular to the line from dl to P in the direction indicated. This field intensity dH may be resolved into two components, one parallel to the axis OP of the coil, and the other perpendicular to OP. Similarly, the current in dl' produces an equal field intensity dH' perpendicular to the line from dl' to P, which may also be resolved into two components, one parallel to and the other perpendicular to OP. The perpendicular components due to dl and dl' are equal; similarly for any other pair of equal elementary lengths diametrically opposite in the circumference of the circle. Hence the resultant field intensity at P is the sum of the components parallel to OP of the field inten-

sities at P due to all the elementary lengths in the circumference of the circle. The component parallel to OP of the field intensity dH due to any element is

$$dH.\ \sin \alpha = \frac{I\,dl}{a^2+r^2} \cdot \frac{r}{\sqrt{a^2+r^2}}$$

which, integrated along the circumference of the circle, gives as the value of the resultant field intensity at P

$$H = \int_{l=o}^{l=2\pi r} \frac{Ir\,dl}{(a^2+r^2)^{\frac{3}{2}}} = \frac{2\,\pi\,r^2\,I}{(a^2+r^2)^{\frac{3}{2}}} \qquad (5)$$

All quantities in this equation are in *c. g. s.* units.

The direction of the field intensity at any point along the axis is the direction in which a right-handed screw placed at this point advances when it is turned in the direction of the current.

The field intensity at the *center* of the coil is found by putting a equal to zero in the above equation, which gives

$$H_c = \frac{2\,\pi\,I}{r} \qquad (5a)$$

The field intensity at any point on the axis of a circular coil which has any number of concentrie turns may also be calculated from equation (5), by calculating the intensity due to each turn separately and adding these separate intensities. In the case of a circular coil with a *concentrated* winding of N turns, *i.e.*, when the N turns are so close together that they may all be considered as occupying but a single geometrical line, the field intensity at the center of the coil is, from (5*a*),

$$H_c = \frac{2\,\pi\,N\,I}{r} \qquad (5b)$$

The winding of a coil may be considered as concentrated when the radius of the coil is large compared with the linear dimensions of the cross section of the winding.

In Article 108 is deduced by a different method the field intensity for a point inside a long coil wound in the form of a long cylindrical helix or "solenoid."

75. Absolute Measurement of an Electric Current. — When a current is established in an insulated wire wound into a circular coil of N turns, the strength of this current can be determined experimentally in terms of quantities which can be measured or

calculated. Let such a coil (Fig. 54) be set up with the plane of its windings parallel to the direction of the earth's magnetic field and let a small magnetic needle be suspended at the center of the coil in such a manner that it is free to turn about a vertical axis. When there is no current in the coil, this needle will then point in a direction parallel to the plane of the coil. When a current is established in the coil, this current will set up a magnetic field at right angles to the plane of the coil, and therefore the direction of the resultant field at the center of the coil will be changed and the needle will therefore be deflected a certain angle θ. This angle will be equal to the angle between the direction of the resultant field and the direction of the horizontal component of the earth's field at the center of the coil. This latter field intensity, which we may call H_e, can be determined by the method described in Article 42. The intensity of the field at the center of the coil H_c is given by equation (5b).

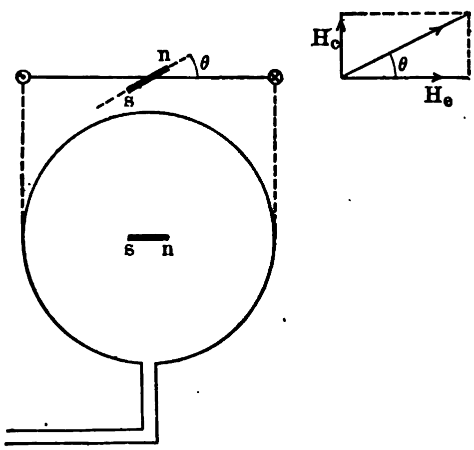


Fig. 54

We then have that

$$\tan\theta = \frac{H_c}{H_e} = \frac{2\pi NI}{rH_e}$$

or

$$I = \frac{rH_e \tan\theta}{2\pi N} \qquad (6)$$

where all the quantities are in *c. g. s.* units.

Since all the quantities in the right-hand member of this equa-

tion can be measured, the strength of the current I can be calculated. Such a device is called a *tangent galvanometer*. The accuracy of the instrument depends upon the accuracy with which the horizontal component of the earth's field may be measured, but an accurate measurement of the latter is difficult. Besides, in any ordinary laboratory or testing room the electric circuits in the building and the surrounding streets also produce magnetic fields which act on the needle, and these fields are continually changing. Such an instrument is therefore seldom used nowadays.

A much more accurate method of determining the absolute value of an electric current is to cause the same current to flow through two parallel coils, one of which is suspended from one arm of a delicate balance. The stationary coil then produces a magnetic field which produces a pull on the movable coil and the amount of this pull can be readily measured. From equations (2) and (3) the amount of this pull in terms of the current and the dimensions of the coils can also be deduced, and since both the pull and the dimensions of the coil can be measured, the strength of the current can be calculated. Such an instrument is called an "absolute" current balance. This is also the principle of the Kelvin current balances; in the latter, however, the current strength is not determined directly from the pull and the dimensions of the coils, but is calculated from the position of a "rider" on an arbitrary scale; the latter is "calibrated" by comparison either with an absolute balance or with a silver voltameter (see Article 79).

76. Comparison of the Strengths of Electric Currents.— Galvanometers and Ammeters. — We have just seen how the strength of an electric current flowing through the coils of a current balance may be measured in terms of the dimensions of the coils and the pull of the fixed coil on the movable coil. The accurate measurement of an electric current by this method requires a balance constructed with great care and numerous precautions have to be taken in using it. Much simpler and cheaper instruments can be constructed which will indicate by the deflection of a needle or spot of light the *relative* magnitude of the electric currents which may be established in them.

The simplest form of such an instrument is a device which consists essentially of a magnet suspended inside a coil of insulated wire which may be connected in series with the conductor in which

the current to be measured is established. The magnetic field produced inside the coil by the current produces a force on the needle and causes it to deflect from its position of equilibrium; this force, and therefore the deflection, varies with the current in the coil. A device of this kind is called a moving needle, or Thomson, galvanometer. The value of the current corresponding to a given deflection can be determined once for all (provided the conditions of operation remain unchanged) by connecting in series with the galvanometer a standard current balance and noting the value of the deflection corresponding to various current strengths as indicated by the standard balance; that is, the galvanometer can be "calibrated" by comparing it directly with the standard balance. In practice, instead of using a standard balance, one ordinarily uses a "secondary" standard, that is, some other instrument which has been previously calibrated by comparison with a "primary" or absolute standard.

The needle galvanometer just described does not "hold" its calibration for any great length of time, on account of the effects of temperature, moisture, etc., on the fiber supporting the magnetic needle and also on account of the variation in the direction and intensity of the earth's magnetic field. A superior form of galvanometer for most practical purposes consists of a coil of fine insulated wire suspended by a metal fiber or thin metal strip, between the two poles of a powerful permanent magnet made in the form of a horseshoe. The fixed end of the metal strip supporting the coil is connected to a suitable terminal or "binding post" on the frame of the instrument and the other end to one end of the wire forming the coil. The other end of the coil is connected through a second metal strip, usually wound in the form of a light spiral spring, to a second terminal or binding post. When the two terminals of such an instrument are connected in series with the circuit in which the current to be measured is established, the same current is also established in the wire forming the movable coil, and the force produced on this coil by the magnetic field due to the permanent magnet causes a deflection of the coil. (See equation 2.) This deflection can be read by noting the deflection of a spot of light reflected from a mirror attached to the coil. This type of galvanometer is called a moving coil, or D'Arsonval, galvanometer, from the name of the physicist who introduced it. A D'Arsonval galvanometer can be calibrated

by comparing it with a standard instrument in the same way that a moving needle, or Thomson, galvanometer is calibrated.

An instrument extensively used for the measurement of electric currents, and known as the Weston ammeter, is essentially a D'Arsonval galvanometer. In this instrument the movable coil, instead of being suspended by a metal strip, is provided with a fine pivot which rests in a jewel bearing. The current is "led to and from" the instrument by means of two small flat spiral springs connected to the two ends of the coil respectively and also to the terminals on the outside of the case of the instrument. These springs also serve to hold the coil normally in a definite position. A light metal pointer is attached to the coil, and the position of the pointer is read off on a scale which is marked to read directly in amperes. The scale of such an instrument is seldom exactly correct, and for accurate work the instrument must be calibrated in the manner described above. There are other makes of ammeters based upon the principle of the D'Arsonval galvanometer, and still others in which the current in the coil of the instrument sets up a magnetic field which exerts a force on a piece of soft iron and thereby causes the deflection of a pointer over a graduated scale. For a detailed description of ammeters, the reader is referred to any text-book on electric measuring instruments.

77. Electrolysis and Electrolytes. — Having now seen how the strength of an electric current may be measured, we are ready to investigate some of the other phenomena which are attributed to the flow of an electric current. An important group of phenomena are those which take place in a conducting solution when a current is established through the solution. In the first place, in any solution which is a conductor, chemical decomposition always takes place. The change that takes place in the copper sulphate solution described in Article 62 is an illustration. Any substance the constituents of which are separated when an electric current is established in it, is called an *electrolyte*, and the process of separation is called *electrolysis*. All conducting liquids other than molten metals are electrolytes; gases, when they become conducting, are also electrolytes. Some solids are also electrolytes, an example of which is silver iodide. The decomposition that takes place in the electrolyte is found to be confined entirely to the portions of

the electrolyte in contact with the metal plates or wires (called the *electrodes*) which connect the electrolyte to the rest of the circuit. The results of the decomposition are therefore usually said to be *deposited* at the electrodes; though, of course, if the substance liberated is a gas, it will immediately escape through the surface of the liquid, or if the substance liberated is soluble, it will go into solution, or if it is a solid which is not soluble, it may fall to the bottom of the vessel containing the electrolyte. In certain cases when the result of the decomposition is a metal, the metal becomes firmly attached to the electrode where it is liberated. The copper deposited on the copper wire in the copper sulphate solution in Article 62 is an example. The important industry of electroplating is based upon this fact.

The electrode through which the current enters the solution is called the *anode*, and the electrode through which the current leaves the solution is called the *cathode*. With very few exceptions, an element, or such a group of elements as is called a " radical," is always deposited at the same electrode, no matter from what compound it may be liberated. Hydrogen and metals are always deposited at the cathode.

As the result of a series of careful experiments Faraday found that a simple relation exists between the rate at which a substance is deposited from an electrolyte, when an electric current is established in it, and the strength of the current. This relation is

1. The rate (mass per unit time) at which a substance is deposited from an electrolyte, when an electric current is established in it, is directly proportional to the strength of the current established.

Faraday also found that a simple relation exists between the masses of the various substances deposited, when the *same* current is established in several electrolytes, and the chemical equivalents* of these substances. This relation is

2. When the same current is established through different

*The chemical equivalent of an element or radical is the atomic weight of the element or radical divided by its valence. By valence is meant the number of atoms of hydrogen with which one atom of the element or radical will form a stable combination. For example, the valence of oxygen is 2, since 2 atoms of hydrogen combine with 1 atom of oxygen to form water. The valence of copper in cuprous compounds is 1 and in cupric compounds 2.

electrolytes the rates (mass per unit time) at which the various substances are deposited are directly proportional to the chemical equivalents of these substances.

These two statements of experimental facts are known as Faraday's Laws of Electrolysis.

78. Electrochemical Equivalent of a Substance. — Faraday's first law may be expressed mathematically by the equation

$$\frac{m}{t} = k\,I \tag{7}$$

where I is the current established in the electrolyte, m the mass of a given substance deposited in time t, and k a constant of proportionality, the value of which depends only upon the nature of the substance and the units in which m, t and I are measured. This constant is called the *electrochemical equivalent* of the substance. Its value for any substance may be readily determined experimentally by measuring the current I by means of an absolute current balance, and measuring the mass of the substance deposited when this current flows through the electrolyte for a given length of time. The values of the electrochemical equivalent for silver, copper, hydrogen and oxygen, when m is measured in grams, t in seconds and I in amperes, are

Silver	0.001118
Copper (from cupric solutions)	0.0003293
Hydrogen	0.00001044
Oxygen	0.0000829

In equation (7) the current is assumed to be continuous; when the current varies with time the mathematical expression of Faraday's first law is

$$\frac{dm}{dt} = ki$$

or the total mass of the substance deposited in time t is

$$m = k\int_0^t i\,dt \tag{7a}$$

79. The International Ampere. — Knowing the value of the electrochemical equivalent of a given substance, for example, silver, one can readily determine the strength of the current flowing through an electrolyte from which this substance is deposited, by measuring the number of grams of this substance deposited

in a measured interval of t seconds. The only apparatus required is a suitable receptacle for holding the electrolyte and suitable electrodes for leading the current in and out. Such a piece of apparatus is called a *voltameter*. From the ease and accuracy with which these measurements can be made, the International Congress of Electricians (Chicago, August 21, 1893) adopted the following as the definition of the ampere:

"As a unit of current (shall be taken), the International Ampere, which is one-tenth of the unit of current of the C. G. S. system of electromagnetic units, and which is represented sufficiently well for practical use by the unvarying current, which, when passed through a solution of nitrate of silver in water, in accordance with the accompanying specification (A), deposits silver at the rate of 0.001118 gramme per second."

The specification A referred to describes in detail the construction of the voltameter and the method of using it. This specification will be found in full in *Foster's Electrical Engineer's Pocket Book*, page 10. The above definition of the ampere has been legalized by most of the civilized countries,* and is therefore sometimes called the "legal" definition of the ampere.

80. Quantity of Electricity. — In the case of a current of water flowing through a pipe, the strength of the water current is defined as the quantity of water which flows across any cross section of the pipe in unit time. In the case of what is called a current of electricity, we have found it convenient to define first the strength of the electric current. From the analogy which is found to exist between the properties which must be attributed to the something called electricity and the properties of an incompressible fluid, the *quantity* Q of electricity flowing across any cross section of a conductor in any time t may be defined as the product of the strength of the current I at this cross section by the time t, that is,

$$Q = It \tag{8}$$

provided the current is a continuous current, *i.e.*, does not vary with time. In case the current does vary with time the general definition of the quantity Q of electricity which flows across any given cross section of the conductor in any interval of time t,

*Germany and Switzerland are exceptions; in these countries the electrochemical definition is taken as the fundamental definition of current strength.

is the integral with respect to time of the variable current at that section, that is

$$Q = \int_{o}^{t} i\,dt \qquad (8a)$$

where i represents the value of the current at any instant, dt an infinitesimal interval of time measured from this instant, and t the total time during which the current flows. From the relation expressed by equation (7*a*) the quantity of electricity which flows through an electrolyte in any interval of time may be determined by measuring the mass of the substance deposited at either electrode in this interval. In Article 109 is described the method usually employed for measuring the quantity of electricity corresponding to a variable current.

Since in the case of a continuous current the total current strength is the same at all cross sections of a given conductor, it follows that the quantity of electricity flowing across each cross section of the conductor in any given interval of time must also be the same at each cross section. Note the analogy with an incompressible fluid. Hence the flow of a current of strength I through the conductor for t seconds may be looked upon as equivalent to the transfer of $I\,t$ units of electricity from one end of the conductor to the other, just as a current of ten cubic feet of water per second in a pipe for five seconds is equivalent to a transfer of 10 x 5 = 50 cubic feet of water from one end of the pipe to the other. In the case of water in a pipe the 50 cubic feet of water which enter the pipe may not be the same as the 50 cubic feet which leave the pipe at the other end. In the same way, it is not necessary that we look upon the electricity which enters one end of a conductor as being the same electricity as leaves the conductor at its other end.

The unit of quantity of electricity in the *c. g. s.* electromagnetic system of units is the quantity of electricity transferred by a current of one abampere for one second. This unit is called the absolute unit of quantity or the *abcoulomb*. The practical unit of quantity of electricity is the *coulomb*, which was defined by the International Congress of Electricians as follows:

"As the unit of quantity (shall be taken), the International Coulomb, which is the quantity of electricity transferred by a current of one international ampere in one second."

The *ampere-hour*, *i.e.*, the quantity of electricity corresponding

to a current of one ampere for one hour is also employed in practice.

In accordance with these definitions we then have

$$1 \text{ abcoulomb} = 10 \text{ coulombs}$$

$$1 \text{ ampere-hour} = 3600 \text{ coulombs}$$

81. Electric Resistance. — Joule's Law. — One of the phenomena always associated with an electric current in a conductor is the dissipation of heat energy in the conductor. In the case of a wire of uniform structure kept at a constant uniform temperature experiment shows that the rate at which heat energy is dissipated in a given length of the wire, between any two points 1 and 2 say, when a continuous current is established in the wire, is directly proportional to the square of the strength of the current in this wire. That is, calling P_h the rate at which heat energy is developed in the wire between the points 1 and 2, and I the strength of the current in this portion of the wire, then

$$P_h = R I^2 \tag{9}$$

where R is a constant depending upon the dimensions and temperature of the wire, the nature of the substance forming the wire, and the units in which P_h and I are measured, but is *independent* of the current strength. This factor R is called the *resistance* of the wire, and the statement of the experimental fact represented by this equation is called Joule's Law, from the name of the scientist who first clearly enunciated the fact. The resistance of a given length of wire may then be defined as *the ratio of the rate at which a continuous current produces heat energy in the wire to the square of the strength of this current.* When the rate P_h at which the heat energy is dissipated is expressed in ergs per second and the current I in abamperes, a resistance of one unit is equal to the ratio of one erg per second to one abampere-squared. This unit of resistance is called the *c. g. s.* electromagnetic unit of resistance or the *abohm.* When the rate P_h at which heat energy is dissipated is expressed in joules per second (one joule by definition is equal to 10^7 ergs) and the current I in amperes, a resistance of one unit is equal to the ratio of one joule per second to one ampere-squared. This unit is called the practical unit of resistance or the *ohm.* The relation between the ohm and the abohm is therefore

$$1 \text{ ohm} = 10^9 \text{ abohms.}$$

To express small resistances a unit one-millionth of the size of an ohm is ordinarily used; this unit is called the *microhm.* To express large resistances a unit one million times the size of an

ohm is frequently used; this unit is called the *megohm*. Hence

$$1 \text{ ohm} \quad = 10^6 \text{ microhms}$$
$$1 \text{ megohm} = 10^6 \text{ ohms}.$$

The resistance of a wire to a variable current is the same as its resistance to a continuous current *provided* the wire is small and the current does not vary rapidly with time. In the case of a large wire the resistance of any current filament (see Article 101) is the same to the variable as to a continuous current, but the resistance of the wire as a whole is greater. The rapid variation of the current with time causes a different distribution of the stream lines of the current and thereby produces a greater heating than that which takes place due to a continuous current. See Article 121.

82. Absolute Measurement of Electric Resistance. — The determination of the electric resistance in terms of the quantities specified in the above definition would require the measurement of the heat energy dissipated in the wire in a given interval of time when a continuous electric current of known strength is established in the wire and the wire kept at constant temperature. The interval of time can be readily measured, and the current strength may be determined directly by an absolute current balance or by means of any kind of galvanometer or ammeter which has been calibrated. The heat energy dissipated in the wire could be determined by some calorimetric measurement. Calorimetric measurements, however, are difficult and at best are not susceptible of a high degree of accuracy. A much more accurate method of measuring the electric resistance of a wire in terms of quantities which may be measured or readily calculated, is the following, which is based upon the principle of electromagnetic induction (see Chapter IV), but the only quantities to be measured are those which have already been defined. A metal disc D, mounted on a metal axis coincident with the axis of a coil of wire C, is arranged so that it can be rotated with a constant angular velocity ω. The coil C and the resistance R to be measured are connected in series with a battery B. A galvanometer G has one of its terminals connected by a wire to the axis of the disc and the other terminal to one end of the resistance R, the other end of R is connected by a wire to a metal "brush" making contact with the rim of the disc. We have then two closed circuits in each of which the resistance R forms a part; one circuit is the

battery, the coil and the resistance R and the connecting wires, and the other circuit is the galvanometer, the disc, the resistance R and the connecting wires. When the disc is set in rotation, it is found that the current through the galvanometer depends upon the angular speed at which the disc is driven (due to the cutting by the disc of the lines of induction produced by the current in the coil, see Chapter IV), and as a result of the two principles known as Kirchhoff's Laws (see Article 98), it can be deduced that when the current in the galvanometer is zero, the relation between

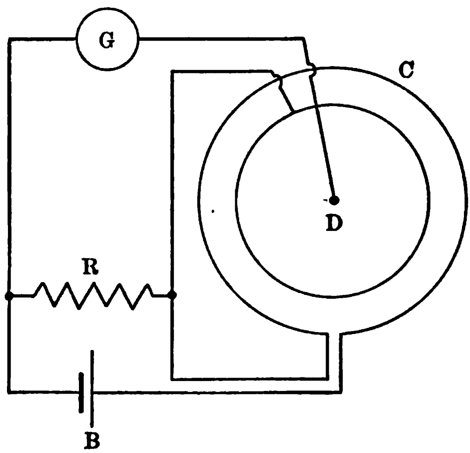


Fig. 55.

the value of the resistance R as above defined, in abohms, and the angular speed in radians per second, must be

$$R = \frac{M\,\omega}{2\,\pi} \tag{10}$$

where M represents the number of lines of induction threading the disc D due to the magnetic field set up by a current of one abampere in the coil C. This quantity M can be calculated in terms of the dimensions of the coil, which can of course be measured, as can also the angular speed ω. This method for determining the value of a resistance is known as Lorenz's Method. There are still other methods for determining experimentally the value of a resistance in terms of quantities which can be actually measured or calculated, some of which are described in J. J. Thomson's *Elements of Electricity and Magnetism*, page 462 ff.

The results of a large number of measurements by these various methods show that a column of mercury 106.3 centimeters long which has a uniform cross section of 1 square millimeter has a resistance of one ohm at zero degrees centigrade. A column of mercury of these dimensions has a mass of 14.4521 grams. Hence the adoption of the following definition of the ohm by the International Congress of Electricians:

"As a unit of resistance (shall be taken), the International Ohm, which is based upon the ohm equal to 10^9 units of resistance of the C. G. S. system of electromagnetic units, and is represented by the resistance offered to an unvarying electric current by a column of mercury at a temperature of melting ice, 14.4521 grammes in mass, of a constant cross sectional area, and of the length 106.3 centimeters."

This definition, like that of the ampere, has been legalized by most civilized countries.

Although the absolute measurement of a resistance is comparatively difficult, the *comparison* of the values of two or more resistances is a very simple matter and can be carried out with a high degree of accuracy. All these methods of comparison are based upon Kirchhoff's Laws. (See Article 92.)

83. Specific Resistance or Resistivity. — As already noted, the resistance of a conductor depends upon its dimensions. It is found by experiment that the resistance of a conductor of uniform cross section throughout its length, when the conductor is kept at a uniform temperature throughout and the current density (see Article 101) is uniform over its cross section, varies directly as the length of the conductor and inversely as the area of its cross section (perpendicular to its length), but does not depend upon the *shape* of its cross section. That is, calling l the length of the conductor, A the area of its cross section, and R the resistance of this conductor, then

$$R = \rho \frac{l}{A} \tag{11}$$

where ρ is a factor of proportionality which depends upon the *material* of which the conductor is made and the *temperature* at which the conductor is kept, and also upon the units in which R, l, and A are expressed. The value of this factor ρ for any conductor, at any temperature, is called the *specific resistance* or the *resistivity* of the conductor at that temperature. When R

is expressed in microhms, l in centimeters and A in square centimeters, this factor ρ is equal to the resistance in microhms of a cube of the conductor 1 cm. on each edge, provided the stream lines of the current are parallel to four parallel edges of the cube and the current density over the section at right angles to these stream lines is uniform (see Article 101). The specific resistance of the conductor may then be expressed as so many microhms per centimeter-cube. Similarly, when R is expressed in microhms, l in inches and A in square inches, this factor ρ is equal to the resistance in microhms of an inch-cube of the conductor; the specific resistance of the conductor may then be expressed as so many microhms per inch-cube. Again, when R is expressed in ohms, l in feet and A in circular mils* this factor ρ is equal to the resistance in ohms of a portion of the conductor one foot long and one circular mil in cross section; the specific resistance of a conductor may then be expressed as so many ohms per mil-foot.

There is still another way of expressing the specific resistance of a conductor, which was formerly much used and is still employed sometimes in wire specifications. The cross section of a bar or wire of uniform cross section is equal to the volume of the bar divided by its length, and the volume of the bar is in turn equal to its mass divided by the density of the material of which it is made. Hence, using the same symbols as above, and in addition calling m the mass of the conductor and δ its density or specific gravity, we have

$$A = \frac{m}{\delta\, l} \text{ and therefore } R = \rho\,\frac{l}{A} = \rho\,\delta\,\frac{l^2}{m}$$

For a given material at a given temperature $\rho\,\delta$ is also constant, hence

$$R = k\,\frac{l^2}{m} \tag{11a}$$

*A *mil* is defined as one-thousandth of an inch and a *circular mil* is defined as the area of a circle one mil in diameter. Since the area of a circle varies as the square of its diameter, the area of a circle in circular mils is equal to the square of its diameter in mils. This unit of area is therefore very convenient for expressing the area of the cross section of a circular wire, since the factor π is eliminated. The area of a circle one-thousandth of an inch in diameter is equal to $\frac{\pi}{4}(0.001)^2$ square inches or to $\frac{\pi}{4}(0.001\times 2.54)^2$ square centimeters; hence

$$\begin{aligned} 1 \text{ circular mil} &= 0.78540\times 10^{-6} \text{ square inches} \\ &= 5.0671 \times 10^{-6} \text{ square centimeters} \end{aligned}$$

where k is a constant (equal to $\rho\,\delta$) for a given material at a given temperature. The specific resistance of a conductor may then also be expressed indirectly in terms of this factor k. When R is expressed in ohms, l in meters and m in grams, this factor k is equal to the resistance of one gram of the conductor made into a wire of uniform cross section and one meter in length; this factor k is then said to be the specific resistance of the conductor in *ohms per meter-gram.* It should be noted that this method of expressing the specific resistance of a conductor involves the density of the conductor, while the other methods do not. The determination of the specific resistance of a conductor in meter-grams, however, does not require a determination of the density, but merely the measurement of the resistance, length and mass of a given wire of the conductor, since $k=\frac{mR}{l^2}$. However, to calculate from the value of k the specific resistance in microhms per centimeter-cube, or per inch-cube, or ohms per mil-foot, does require a knowledge of the density of the conductor.

The various units of specific resistance are related as follows:

1 microhm per inch-cube	$=2.5400$ microhms per centimeter-cube
1 microhm per centimeter-cube	$=6.0153$ ohms per mil-foot
1 ohm per meter-gram	$=\frac{100}{\delta}$ microhms per centimeter-cube
1 ohm per meter-gram	$=\frac{601.53}{\delta}$ ohms per mil-foot

To calculate the resistance of a wire of uniform cross section when its specific resistance is given, the formulas

$$R=\rho\frac{l}{A} \quad \text{or} \quad R=k\frac{l^2}{m}$$

are directly applicable, but care must be employed to express all the quantities entering into the formula in the *same system* of units. For example, the resistance in *ohms* of a wire which has a specific resistance of 1.6 microhms per centimeter cube, a length of 1000 feet and a cross section of $\frac{1}{4}$ square inch, is

$$R=1.6\times10^{-6}\,\frac{1000\times12\times2.54}{0.25\times(2.54)^2}=0.0302\text{ ohms.}$$

Experiment shows that the resistance of a given length of wire of uniform cross section is independent of the shape into

which the wire is bent, provided the diameter of the wire is small compared to the radius of curvature of the curve into which it may be bent, which condition is equivalent to saying that the "stream lines" of the current are all of the same length (see Article 101). This condition is almost always realized in practice, and consequently formulas (11) and (11*a*) are in general directly applicable to the calculation of the resistance of a wire whether the wire be straight or curved or wound into a coil of any shape. These formulas are also applicable to the calculation of the resistance of a rod or bar, provided the rod or bar is not bent into a sharp curve and the distance between its points of connection to the circuit is large compared to the linear dimensions of its cross section. When the stream lines of the current are not all of the same length, as, for example, when a current is established in a heavy short bar bent into a sharp curve, the resistance of the bar can be calculated only when the distribution of these stream lines is known. Again, when the stream lines of the current are not parallel, *e.g.*, the leakage current through the insulation of a cable, these formulas are not applicable (see Article 102).

With the exception of silver, copper has the lowest specific resistance of any metal. The specific resistances of a few common metals at 0° centigrade are given below.

	Microhms per Cm.-Cube	Ohms per Mil-Foot	Ohms per Meter-Gram
Silver	1.49	8.94	0.156
Copper (very pure, annealed)	1.56	9.35	0.139
Aluminum (99% pure)	2.56	15.4	0.067
Iron (very pure)	9.07	54.5	0.707
Steel rails (average*)	14.00	84.2	1.09

Silver is seldom used for electric conductors on account of its high cost. Copper is the metal most frequently used. Aluminum, although it has a greater resistance than copper for the same length and cross section (*i.e.*, same volume), has, on account of its low density, a less resistance for the same mass or weight; its cost for the same resistance as that of an equal length of copper is about 10% less. It is therefore used to a considerable extent for overhead transmission lines; its mechanical strength, however, is inferior to that of copper. Pure iron is too costly for use as a

*The resistance of steel rails varies considerably with their chemical composition.

practical conductor. Steel, however, in the form of track rails or "third" rails is largely used in railway work, but is seldom used as a conductor in the form of wires, except for short telephone lines, and even here it is being largely supplanted by copper.

Any impurity in a metal increases its resistance. All alloys have a greater resistance than that of the best conductor in them. For certain purposes, particularly in the construction of "resistance boxes" and "rheostats," a high specific resistance is desirable. Various alloys are made for this purpose, the specific resistances of which can be found in any electrical engineer's handbook. For the resistance of copper and aluminum wires of various sizes see Appendix B.

84. Electric Conductance and Conductivity. — The reciprocal of electric resistance is defined as electric conductance. That is, when R is the resistance of a conductor, the conductance is

$$G=\frac{1}{R}=\frac{I^2}{P_h} \qquad (12)$$

where I is the continuous current established in the conductor and P_h the rate at which heat energy is developed in it. The unit of conductance on the *c. g. s.* electromagnetic system of units is one abampere-squared per erg per second; this unit is called the absolute unit of conductance or the *abmho*. The practical unit of conductance is one ampere-squared per joule per second, or one ampere-squared per watt; this unit is called the *mho*. (Note that the word "mho" is simply the word ohm written backwards.)

The reciprocal of specific resistance or resistivity is called *specific conductance* or *conductivity*. For example, a specific resistance of 10 ohms per mil-foot is the same as a conductivity of 0.1 mhos per mil-foot.

85. Matthiessen's Standard of Conductivity. — Matthiessen, about 50 years ago, found that the specific resistance of the purest copper at that time obtainable was 0.141729 ohms per meter-gram at 0° cent. Matthiessen, however, failed to state the density of his copper, so that the exact value of this specific resistance in microhms per centimeter-cube or in ohms per mil-foot is not known. However, assuming 8.89 as the probable value of this density, Matthiessen's value of 0.141729 ohms per meter-gram is equivalent to 9.5900 ohms per mil-foot. The corresponding value of the conductivity, namely 0.10427 mhos per mil-foot, has been adopted by electrical engineers in this coun-

try and in England as the "standard conductivity"; this standard is usually referred to as *Matthiessen's standard.* The *relative conductivity* of any conductor is then equal to the percentage ratio of its conductivity per mil-foot at 0° centigrade to the conductivity (0.10427) of Matthiessen's standard; that is, the relative conductivity of a conductor which has a resistance of 15.5 ohms per mil-foot at 0° centigrade (*i.e.*, 0.0647 mhos per mil-foot) is 62%. Note that the lower the relative conductivity the higher is the specific resistance. It is of interest to note that copper can be made at present of a greater purity than that used by Matthiessen and consequently having a relative conductivity greater than 100% of Matthiessen's standard. Commercial copper wire usually has a conductivity of from 96% to 99%; the "harder" the wire the lower its conductivity. Commercial aluminum has a conductivity of about 62%.

86. Temperature Coefficient of Electric Resistance. — It is found by experiment that the variation of the resistance of a conductor with temperature may be expressed by the formula

$$R = R_0 (1 + \beta t) \qquad (13)$$

where R_0 is the resistance of the conductor at any given "standard" temperature, R the resistance of the conductor at a temperature t degrees above this temperature, and β a coefficient which is approximately constant independent of the temperature rise t, but does depend upon the temperature corresponding to R_0. Zero degrees centigrade is usually taken as the standard temperature, and the value of this coefficient β when the rise of temperature is referred to 0° cent. is called the *resistivity temperature coefficient* of the conductor, or briefly, the *temperature coefficient* of the conductor. The value of β for commercially pure copper depends somewhat upon the purity of the metal and also upon the rise of temperature t. The American Institute of Electrical Engineers, however, has adopted the constant value 0.0042 as sufficiently accurate for all grades of commercially pure copper and for any value of the temperature rise ordinarily met with in electric machinery, that is, for any rise of temperature not greatly in excess of 100° cent. The temperature coefficient of any other commercially pure metal, except the magnetic metals such as iron, nickel, cobalt and bismuth, has practically the same value. The temperature coefficient for iron is 0.00625 per degree cent.

The temperature coefficient of carbon is negative and is not a constant. For example, the resistance of a carbon-filament lamp is less when the lamp is burning than when it is cold. The temperature coefficient of most insulators is also negative and very large; moreover, it is far from being constant.

The temperature coefficient of alloys depends largely upon their constituents. It is possible to make alloys for which the temperature coefficient is zero over a considerable range of temperature. Such alloys are extremely useful in the construction of standard resistance coils, since the resistance of such a standard remains constant for any ordinary variation of temperature and consequently its resistance does not have to be "corrected" for temperature.

Equation (13) enables one to calculate the resistance of a conductor at any temperature when its resistance at any other temperature and its temperature coefficient are known. For, calling R_t the resistance of the conductor at $t°$ cent. and $R_{t'}$ its resistance at t' degrees centigrade, we have

$$R_t = R_o\ (1 + \beta t)$$
$$R_{t'} = R_o\ (1 + \beta t')$$

Hence, taking the ratio of these two equations, we have

$$\frac{R_{t'}}{R_t} = \frac{1 + \beta t'}{1 + \beta t}$$

or

$$R_{t'} = \frac{1 + \beta t'}{1 + \beta t} \cdot R_t \tag{13a}$$

Substituting for β the numerical value of 0.0042, and dividing the numerator and denominator of the fraction in the right-hand member by 0.0042, gives

$$R_{t'} = \frac{238 + t'}{238 + t} . R_t \tag{13b}$$

The operation expressed by this equation is readily performed by one setting of a slide rule, when a temperature scale is marked off on the lower scale of the slide, marking 238 as 0°, 248 as 10°, 258 as 20°, etc. Then, if the resistance at 25° cent. is 3 ohms, say, the resistance at any other temperature, say 60°, is found by setting 25 on this new scale opposite 3 on the lower scale of the rule and reading off on the lower scale the number corresponding to 60 on the new scale, *i.e.*, 3.4 ohms.

Equation (13*b*) also gives a convenient method of determining the average temperature of a coil of copper wire when heated in any manner, as, for example, by the current established in it. For the resistance of the coil "cold," *i.e.*, at room temperature, may be measured and also its resistance when heated. Knowing these two resistances R_t and $R_{t'}'$ and the room temperature t, the value of t' is readily calculated. For example, if the resistance of a coil at 20° cent. is found to be 5 ohms and its resistance when heated 6 ohms, then when 20 on the temperature scale of the slide rule is set opposite 5 on the lower scale, the reading on the temperature scale opposite 6 on the lower scale gives the average temperature, 71° cent. The rise of temperature due to the heating of the coil is then $71-20°=51°$. This method of measuring the temperature rise of a coil is largely used in practice.

87. Difference of Electric Potential.— Electric Energy. — When a steady current of water is forced through a pipe, the work W done in any interval of time by this current in any length of the pipe (between the points 1 and 2, say) is equal to the drop in pressure V from 1 to 2, multiplied by the quantity of water Q which is forced across each section of the pipe between these two points in this interval, *i.e.*,

$$W=VQ$$

or

$$V=\frac{W}{Q}$$

In other words, the drop in pressure from 1 to 2 is equal to the work done by, or the potential energy lost by, the current of water between the points 1 and 2 per unit quantity of water forced through the pipe.

The drop of pressure between the two points may also be expressed in terms of the quantity of water per second forced across each section of the pipe between these two points, *i.e.*, the *strength* I of the water current, and the rate at which work is done by this current, *i.e.*, the *power* P developed by the current. For the quantity of water forced across each section of the pipe in an interval of time t is $Q=It$ and the work done is $W=Pdt$. Whence, from the above expression for the drop of pressure

$$V=\frac{P}{I}$$

That is, the drop of pressure between any two points along a pipe through which a water current is flowing is equal to ratio of the power developed by the current to the strength of the current. This relation holds for a varying current as well as for a steady current; *i.e.*, the instantaneous pressure drop is equal to the instantaneous power divided by the instantaneous value of the current strength.

When there is no pump or motor connected in this given length of pipe, all the work done by the water current appears as heat energy, that is, the potential energy lost by the water in this length of pipe is converted into heat energy. When the water in going from 1 to 2 passes through a water-motor, part of the potential energy lost by the water is converted into mechanical energy by the motor. Hence, when the same quantity of water per second as before is forced through the pipe, the water between the points 1 and 2 will lose potential energy at a greater rate, and consequently the drop of pressure between these points will be greater than before. Again, the water in going through a pump has work done on it, and consequently the water in passing through the pump has its potential energy *increased*. Hence the drop in pressure through the pump in the direction of the flow of water is *negative*, that is, there is a *rise* of pressure through the pump in the direction of the current. In general, whenever the water does work, or loses potential energy, there is a drop of pressure in the direction of the current, and whenever work is done on the water, or the water gains potential energy, there is a rise of pressure in the direction of the current.

In any given system of pipes connecting any number of pumps and motors, in which the pipes, pumps and motors are all completely filled with water, the rate at which work is done by the water in driving the motors, in producing heat energy in the pipes and the water passages of the pumps and motors, and in accelerating itself, is exactly equal to the rate at which work is done on the water by the pumps. That is, the total amount of potential energy of the water remains unaltered; the gain in potential energy by any particle of water as it is forced through a pump is exactly equal to the simultaneous loss of potential energy by some other particle of water somewhere else in the system. The water current then acts simply as a means for the transfer of energy; the work

done on the water at each instant is exactly equal to the work done by it at that instant.

Since the pressure in a current of water has but a single value at each point in this current, it follows that the total drop of pressure around any closed path formed by any number of such pipes, pumps and motors must be zero. Also, since the water is incompressible (at least, practically so) the quantity or volume of water flowing up to any junction of two or more pipes in any interval of time must be exactly equal to the quantity of water which flows away from this junction in this same interval of time. Or, calling the quantity of water which flows *away from* any given point equivalent to an equal *negative* quantity flowing *to* that point, an equivalent statement of this same fact is that the *algebraic* sum of the strengths of the water currents flowing *up to* any junction of the pipes is always zero.

Similarly, it is found by experiment that whenever there occur any of the phenomena which are attributed to the flow of an electric current, there is always a loss of energy by one or more parts of the circuit or by some body in the vicinity of the circuit, and a gain of energy by other parts of the circuit or by bodies in the vicinity of the circuit. For example, in the simple circuit formed by a wire connecting the poles of a battery, the battery loses chemical energy, and the wire and the conductors forming the battery become heated, or gain heat energy. The energy lost by any part of the circuit or by any body in its vicinity is said to result in a gain of an equal amount of electric energy by the electric current, and the energy gained by any part of the circuit or by any body in its vicinity is said to be due to the loss of an equal amount of electric energy by the electric current. That is, the *electric energy gained by the current* in any portion of the circuit is *defined* as the amount of energy lost by this part of the circuit or by any other bodies as the result of the existence of the current in this part of the circuit; and the *electric energy lost by the current* in any portion of the circuit is *defined* as the amount of energy gained by this portion of the circuit or by any other bodies as the result of the existence of the current in this portion of the circuit.*

*Electric energy as thus defined is to be distinguished from electrostatic energy; electric energy is the work done on or by electric *currents*, while electrostatic energy is energy possessed by electric *charges*. Electrostatic energy is discussed in detail in Chapter V.

Experiment shows that the net electric energy, as thus defined, gained by the electric current in any closed electric circuit in any interval of time is *zero*. (This is also true for any stream line of electric current, whether the current be continuous or variable.) That is, any electric energy *gained* by the current in any part of its path, *e.g.*, the energy lost by a battery or other source of electromotive force, is *lost* by the current in some other part of its path, *e.g.*, as heat energy, mechanical energy, chemical energy, or in the case of a variable current, as magnetic and electrostatic energy. The electric energy of an electric current is entirely analogous to the potential energy of a current of water in a pipe or system of pipes completely filled with water. The potential energy gained by the water in any part of its path, *e.g.*, from a pump, is all given out by the water in some other form of energy, *e.g.*, the heat energy developed in the pipe and the work done on any hydraulic motor that may be connected in the pipe. An electric current may then be considered as a means for the transfer of energy, just as a stream of water completely filling a closed pipe or system of pipes is a means for the transfer of energy. The net amount of potential energy in the water current remains constant; similarly the net amount of electric energy in the electric current remains constant.

We have seen how the rate at which potential energy is lost or gained by a current of water flowing through a given length of pipe (*i.e.*, the *power* developed by this current) may be expressed in terms of the strength of the current and the drop of pressure along the given length of pipe. We are therefore led to the conception of *electric pressure*, or, as it is also called, *electric potential*, as a property of an electric current analogous to hydraulic pressure. As in the case of the flow of water, we are concerned only with the difference of pressure and not with the absolute pressure of the water, so in discussing the flow of electric currents we need concern ourselves only with the difference of electric pressure or difference of electric potential. We may then *define* the *drop of electric pressure*, or the drop of *electric potential*, from any point 1 to any other point 2 along a wire carrying an electric current as the ratio of the rate at which electric energy is lost by the current between these two points to the value of the current from 1 to 2. That is, calling P the rate at which electric energy is *lost* by the current between the points 1 and 2 (*i.e.*, the power developed by this current) and I the value of the current from 1 to 2, the

drop of electric potential from the point 1 to the point 2 is defined as

$$V = \frac{P}{I} \tag{14}$$

which is a relation of exactly the same form as that between hydraulic pressure, the rate at which potential energy is lost by the current of water through the pipe, and the quantity of water per second flowing through the pipe. Drop of electric potential, or potential difference, is frequently abbreviated "*p.d.*"

The unit of electric potential drop, or of potential difference, in the *c. g. s.* electromagnetic system of units is equal to one erg per second per abampere and is called the *abvolt*. When the rate at which energy is lost by the current is expressed in joules per second (*i.e.*, watts) and the electric current in amperes, the unit of potential drop is one joule per second (*i.e.*, one watt) per ampere; this unit is called the *volt*, and is the unit almost invariably used in practice. There is still another unit of potential difference which is employed in discussing electrostatic phenomena. This unit is called the *c. g. s.* electrostatic unit. The relations between these various units are

1 volt	$=10^8$ abvolts
1 *c. g. s.* electrostatic unit	$=300$ volts
1 *c. g. s.* electrostatic unit	$=3\times10^{10}$ abvolts

On the hypothesis that an electric current is an actual flow of an incompressible something, the above definition of drop of electric potential is equivalent to the definition that "the drop of electric potential from any point 1 to any point 2 is the work done by unit quantity of electricity when it moves from the point 1 to the point 2." The definition given above, however, is preferable, since it does not involve any hypothesis.

In designating the two terminals of any portion of a circuit that terminal which is at the *higher* potential is called the *positive* terminal and the terminal at the lower potential the negative terminal.

When a potential difference is expressed in volts, the word "voltage" is commonly employed instead of the term potential difference. The word "tension" is also employed to mean the same thing as potential difference. For example, one speaks of the voltage of a generator, a high tension transformer, etc.

88. Measurement of Drop of Electric Potential. — Experiment shows that the drop of electric potential as defined by equation (14) is strictly analogous to the hydraulic pressure in a system of pipes, pumps and motors completely filled with water. In particular, it is found that the drop of electric potential around any closed circuit is always zero, whether this circuit be a simple circuit like that formed by a battery with its poles connected by a wire, or whether this circuit be part of any network of circuits, no matter how complicated the network may be. Hence, when a wire is connected between any two points 1 and 2 of any circuit whatever, the drop of potential from the point 1 to the point 2 through the wire will be exactly equal to the drop of potential from the point 1 to the point 2 through the conductors forming the original circuit. In general, when a wire is connected to an electric circuit, the current established in the wire will cause a change in the strength of the current in the original circuit, which will in turn cause a change in the drop of potential between the two points. Hence, in general, the drop of potential in a wire when it is connected to any two points of a circuit is not the drop which *originally* existed between these two points. However, by making the resistance of the wire sufficiently great, the current which is established in it can be made negligibly small, and the change produced in the current in the original circuit by the presence of the wire may be neglected. Again, (see Article 94) when any two dissimilar substances are placed in contact a small difference of potential is produced between them. Hence if the wire is made of a different material from that of the conductors forming the circuit, the difference of potential between the two ends of the wire will not necessarily be equal to the difference of potential between the two points of the conductor in contact with which the two ends of the wire are placed. This difference, however, is inappreciable in ordinary practical measurements.

The fact that the drop of electric potential between any two points is the same over any path connecting these points, leads to a simple method of actually measuring the potential drop between any two points of an electric circuit in which is established a continuous current. For, experiment shows that when a continuous current is established in a wire which is of uniform structure kept at uniform temperature in an unvarying magnetic field, then

the only form of energy produced in or around the wire is heat energy. Hence from the definition of electric energy given above, the total loss of electric energy by the current in this wire is equal to the heat energy which is produced in the wire. But by Joule's Law (Article 81), the rate P_h at which heat energy is developed between any two points 1 and 2 of such a wire under these conditions, is equal to RI^2, where R is the resistance of the wire between the points 1 and 2, and I is the current in the wire. Hence the drop of potential from the point 1 to the point 2 is, from the definition given above (equation 14),

$$V = \frac{RI^2}{I} = RI. \tag{15}$$

That is, the drop of potential between any two points 1 and 2 in the direction of the current in a wire of uniform structure and of uniform temperature throughout, when this wire is kept in an unvarying magnetic field, is equal to the product of the resistance of the wire between these two points by the strength of the current in the wire; this product RI is frequently called the "resistance drop" between the two points. We have already seen how the current strength and the resistance may be measured; consequently the above relation gives a method for actually measuring the drop of electric potential between two points of such a wire.

To measure the drop of potential between *any* two points 1 and 2 of *any* circuit, it is then only necessary to connect to these points a wire of known resistance and measure the strength of the current established in it. Then, if the resistance of the wire is sufficiently high to make the current in the wire negligibly small, the difference of potential between the two points is equal to the product of the resistance of the wire by the current established in it. This is the principle of the ordinary continuous current voltmeter, which is simply an ammeter with a high resistance coil in series with it. The scale of the instrument is calibrated to read directly in volts instead of in amperes; that is, the scale reads directly the product of the strength of the current through the wire forming the coils of the instrument and the high resistance coil in series with it and the value of the total resistance of this wire. For accurate measurements with such an instrument the effect of changes of temperature in changing the resistance of the coils has to be taken into account. Also, care must be taken not to place the instrument in a strong magnetic field, since

such a field may alter the magnetic field in which the coil of the instrument is designed to move. The effect of using wires of different materials in the various parts of the instrument, or for the connections to the instrument, is usually inappreciable in ordinary practical measurements.

89. Ohm's Law. — Equation (15) may also be expressed in the form

$$R = \frac{V}{I} \tag{16}$$

and in this form is known as *Ohm's Law,* from the name of the scientist who discovered the relation expressed by this formula. In words, Ohm's Law is that, when a continuous current is established in a wire of uniform structure throughout, kept at a constant temperature in an unvarying magnetic field, then the ratio of the potential drop between any two points along this wire to the strength of the current established in the wire is *constant,* independent of the strength of the current.

90. Electric Power and Electric Energy. — From the definition of drop of electric potential given in Article 87, it follows that the rate P at which electric energy is lost by an electric current of strength I between any two points 1 and 2 of an electric circuit is equal to the drop in potential V from the point 1 to the point 2 multiplied by the value of the current from 1 to 2, that is

$$P = VI \tag{17}$$

When an electric current loses electric energy it is said " to develop an amount of power " equal to the rate at which it loses energy. Hence the power developed by an electric current in any portion of an electric circuit is equal to the product of the strength of the current in this portion of the circuit by the drop of potential in this portion of the circuit.

When the current I is measured in abamperes and the potential drop V in abvolts, the product VI gives the power in ergs per second. When I is measured in amperes and V in volts the product VI gives the power in watts. Large amounts of electric power are usually expressed in kilowatts, *i.e.*, thousands of watts. See Article 22 for the relation between the various units of power.

Since both the current and the potential drop may be readily measured, the amount of power developed by a current can be readily determined; in fact, electric power can be measured with

much greater accuracy than any other kind of power. It should be noted that an electric current can develop power only as the result of a loss of an exactly equal amount of power by some device associated with the electric circuit; that is, there must always be a generator of electric power somewhere in the circuit, *e.g.*, a battery or a dynamo driven by some external source of energy.

The relation expressed by equation (17) holds not only for a continuous current but also for the instantaneous values of the quantities involved, irrespective of how these quantities may vary with time.

The amount of work done by the current in any portion of a circuit in which the potential drop in the direction of the current is V, when the current is continuous for a time t and has a strength I, is

$$W = VIt \tag{18}$$

When the potential drop and the current vary with time, the total amount of work done by the current in the time t is

$$W = \int_0^t vidt \tag{18a}$$

When the voltage, current and time are expressed in abvolts, abamperes and seconds respectively equations (18) give the energy in ergs; when these quantities are expressed in volts, amperes, and seconds respectively these formulas give the energy in joules or watt-seconds; when these quantities are expressed in volts, amperes and hours respectively, these formulas give the energy in watt-hours. Large amounts of energy are expressed in kilowatt-hours, *i.e.*, thousands of watt-hours. For the relation between the various units of energy see Article 21.

91. The Wattmeter. — Instead of measuring the current and the potential drop separately by two different instruments, it is possible to measure the value of the product VI directly by means of a single instrument. Such an instrument is called an *electro-dynamometer* or *wattmeter*. This instrument consists essentially of two coils, one of which is stationary and the other mounted inside the fixed coil on a suitable suspension or pivot in such a manner that the planes of the two coils are vertical and the movable coil can turn about a vertical axis. One coil C_v, called the "voltage coil," is connected in parallel with, or "shunted" across, the terminals 1 and 2 of the circuit in which it is desired to

measure the power, and the other coil C_i, called the "current coil," is connected in series with the circuit. The current coil is usually stationary and the voltage coil is movable. A high resistance R is connected in series with the voltage coil, so that only a very small current is established through this coil; this current then depends only upon the difference of potential between the points 1 and 2. The current in the current coil, which is the same as the current in the line, sets up a magnetic field the strength of which is proportional to the strength of this current (see Article 67); this magnetic field produces a moment or

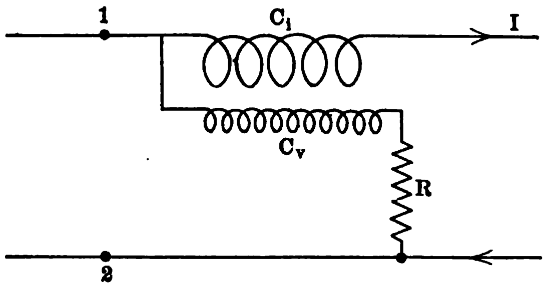


Fig. 56.

torque on the voltage coil, which torque is proportional to the strength of this magnetic field and to the strength of the current established in the voltage coil. Hence the torque produced on the movable coil is proportional to the product of the difference of potential from 1 to 2 and the current in the circuit between these two points. If the opposing torque produced by the suspension of the movable coil, or by a suitable spring attached to it, is proportional to the angular twist given the coil, its deflection will then be proportional to the power transferred to the circuit between these two points. A suitable pointer attached to the movable coil and arranged to move over a suitably graduated scale will then indicate directly the value of the power.

92. Electromotive Force. — The heat energy developed in an electric circuit due to the resistance of the conductors forming the circuit, that is, the heat energy expressed quantitatively by the formula $W_h = RI^2t$, is only one of a number of forms of energy into which electric energy is converted. For example, electric energy may manifest itself by producing chemical changes, that is, by being converted into chemical energy; or by producing a mag-

netic field, which in turn can produce mechanical motion, that is, electric energy may be converted into magnetic energy which in turn is converted into mechanical energy, etc. Again, whenever an electric current is established some body or bodies lose some form of energy, which, in accordance with the definition of electric energy, is converted into electric energy. For example, when a wire is connected to the two poles of a battery, chemical changes take place in the battery which result in a loss of chemical energy by the battery; that is, the chemical energy of the battery is converted into electric energy. Again, it is found by experiment that work is required to change the number of lines of magnetic induction linking any part of an electric circuit; the work which is thus done is, by the definition of electric energy, converted into electric energy.

In any portion of a circuit in which work is done on an electric current or in which the current does work, other than that done as a consequence of the resistance of this portion of the circuit, there is said to exist an *electromotive force.* An electromotive force may then be looked upon as that which produces or opposes the flow of electricity, other than the opposition due to the resistance of the conductor in which it flows. As the measure of the electromotive force in any portion of a circuit is taken the *rise* of potential which it would produce in the direction of the current in this portion of the circuit were there no resistance drop in this portion of the circuit. As noted in Article 88, the resistance drop is always in the direction of the current; hence in any portion of a circuit in which there is a resistance drop the electromotive force is equal to the *resultant rise* of potential in the direction of the current in this portion of the circuit plus *algebraically* the resistance drop in this portion of the circuit. An electromotive force is then considered *positive* with respect to the current when it produces a *rise* of potential in the *same* direction as that of the current; *negative* when it produces a *rise* of potential in the *opposite* direction to that of the current. An electromotive force in the opposite direction to that of the current is frequently called a *back* or *counter* electromotive force. From the definition of potential drop (Article 87) it follows that wherever the current and the electromotive force are in the same direction there is a gain of electric energy by the current, or the current has work done on it; wherever there is a back electromotive force the current loses electric

energy, or does work. Electromotive force is frequently abbreviated *e. m. f.*

An electromotive force is analogous to the pressure developed by a pump, while a back electromotive force is analogous to the back pressure due to hydrostatic head or to the back pressure due to an hydraulic motor. The resistance drop in a conductor is analogous to the drop of pressure, or "loss of head," due to the friction of a pipe. When the terminals of any device which develops an electromotive force are connected by a conductor, the electromotive force may be looked upon as the cause of the flow of the electric current, just as the pressure developed by a pump may be looked upon as the cause of the flow of the water current through a pipe connecting its outlet and intake. Just as the pressure developed by the pump is equal to the drop in pressure in the pipe and the water passages of the pump due to their frictional resistance (provided there is no acceleration of the water or other source of back pressure in the pipe), so is the electromotive force developed by any device equal to the drop of electric pressure, or potential drop, due to the electric resistance of the conductors forming the electric circuit (provided the electric current is continuous, *i.e.*, does not vary with time, and there is no back electromotive force in the circuit). Again, when the outlet and intake of a pump are connected respectively to the intake and outlet of an hydraulic motor, the pressure developed by the pump is no longer equal to the drop in pressure due to the frictional resistance of the pipe and water passages of the pump and motor, but is equal to this drop *plus* the back pressure due to the motor; similarly, when any device which develops an electromotive force is connected by conductors to another device which develops a back electromotive force (*e.g.*, an electric motor) the electromotive force of the first device is not equal to the resistance drop in this circuit, but is equal to this drop *plus* the back electromotive force developed by the second device.

Since an electromotive force is measured by the difference of potential it produces, the units of electromotive force are the same as those of potential difference, namely, the abvolt, volt, and the *c. g. s.* electrostatic unit (see Article 87).

93. Generalized Ohm's Law.—The relation between current strength, electromotive force, potential drop, and resistance in any portion of an electric circuit in which a continuous electric current

is established may be stated in a comprehensive manner in a single formula. Fig. 57 represents diagrammatically any portion of an electric circuit; the terminals of this circuit are designated by the numbers 1 and 2. An electromotive force developed in this circuit in the direction from 1 to 2 is designated by the symbol E_{12} and an electromotive force in the opposite direction by the symbol E_{21}. Similarly, a current in the direction from 1 to 2 is designated by I_{12} and a current in the opposite direction by I_{21}. The drop of potential from 1 to 2 is designated by V_{12} and a drop of potential from 2 to 1 is designated by V_{21}. The resistance of the conductors forming this part of the circuit to the current from one terminal to the other is represented by the symbol R; this resistance is, of course, independent of the direction of the circuit. Then, since E_{12} represents a *rise* of potential in the direction from 1 to 2, and RI_{12} is a *drop* of potential from 1 to 2, the net *rise* of potential from 1 to 2 is $E_{12} - RI_{12}$. But the net *rise* of potential *from* 1 *to* 2 is equal to the net *drop* of potential *from* 2 *to* 1; hence

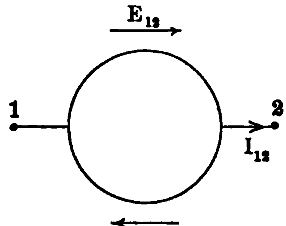


Fig. 57.

$$V_{21} = E_{12} - RI_{12}$$

or

$$I_{12} = \frac{E_{12} - V_{21}}{R} \qquad (18a)$$

The net *rise* of potential from one terminal to another of any portion of an electric circuit is frequently called the *terminal* electromotive force of this portion of the circuit, or the electromotive force *impressed* upon this portion of the circuit by the rest of the circuit. Hence, calling E'_{12} the terminal or impressed electromotive force in the direction from 1 to 2, we have that $E'_{12} = -V_{12} = V_{21}$, and therefore equation (18*a*) may be written

$$I_{12} = \frac{E_{12} - E'_{12}}{R} \qquad (18b)$$

Equations (18*a*) and (18*b*) are two ways of expressing the *same* fact; they hold, of course, only when the current entering one terminal is the same as the current leaving the other.

Equation (18*b*) is frequently called the Generalized Ohm's Law, since this expression reduces to the same form as the expression for Ohm's Law (equation 16) when no electromotive force is developed

in the circuit, *i.e.*, when $E_{12}=0$ then $I_{21}=\frac{E'_{12}}{R}=\frac{V_{21}}{R}$. Equation (18*b*) is an extremely useful one, since it gives a simple and entirely general formula for calculating the current in any portion of a circuit when the electromotive force developed in this portion of the circuit, the impressed or terminal electromotive force, and the resistance of this portion of the circuit are known. In applying this formula care must be taken in not confusing the electromotive force *developed in* the given portion of the circuit with the electromotive force *impressed* across its terminals.

When the generated electromotive force E and the current I are in the *same* direction, *i.e.*, in an electric generator or in any other device in which electric energy is generated, equation (18*b*) may be written

$$E'=E-RI \tag{18c}$$

In this case the terminal electromotive E' force is always less than the generated electromotive force E. When the generated electromotive force E and the current I are in opposite directions, *i.e.*, in an electric motor or in any other device which absorbs electric energy, equation (18*b*) may be written

$$E'=E+RI \tag{18d}$$

In this case the impressed electromotive force E' is greater than the back electromotive force E.

For example, a continuous current dynamo may be used either as a generator or a motor. If the electromotive force generated by the machine is 110 volts in each case, and the resistance of its armature is 1 ohm, then when a current of 10 amperes is supplied by the dynamo running as a generator the terminal electromotive force is $110-(1\times10)=100$ volts; while if the dynamo is running as a motor the impressed electromotive force must be $110+(1\times10)=120$ volts.

In employing the special forms (18*c*) and (18*d*) of the general equation (18*b*), the relative directions of the various quantities must not be lost sight of. In the generator equation (18*c*) the generated electromotive force E, the terminal electromotive force E' and the current I are all in the same direction. In the motor equation (18*d*) the generated electromotive force E and the current I are in opposite directions; the impressed electromotive force E', considered as localized in the rest of the circuit connecting the terminals of the portion of the circuit under consideration, acts

around the closed circuit of which the given portion is a part in the *same* direction as the current, and therefore in the *opposite* direction around the circuit to the back electromotive force E.

The student should bear in mind that the various formulas given in this article are all simply different ways of expressing a single relation. This relation is that the current in any portion of a circuit formed by one or more conductors in series is equal to the *algebraic* difference between the generated and impressed electromotive forces divided by the resistance of the given portion of the circuit.

94. Contact Electromotive Force. — It is found by experiment that whenever an electric current is established in two or more conductors connected in series, the current always gains or loses electric energy at each point of contact between *dissimilar* conductors. For any given point of contact, it is found that whether there is a gain or loss of energy depends upon the direction of the current with respect to the two conductors. For example, when a copper wire is connected to an iron wire and a current is established across the junction of these two conductors, the wires in the vicinity of the junction become cooled when the current is in the direction from the copper to the iron, while if the current is in the opposite direction, from the iron to the copper, the wires in the vicinity of the junction become heated. These effects as a rule are scarcely appreciable, but when special precautions are taken they can readily be detected. In the first case, then, the current gains electric energy at the junction, and in the second case it loses electric energy at the junction.

In general, then, at the junction between any two dissimilar conductors there is an *electromotive force*, and the direction of this electromotive force is independent of the direction of the current. Experiment also shows that the value of this *electromotive force of contact*, as it is called, does not depend upon the strength of the current or upon the area or shape of the surface of contact between the two conductors, but depends only upon the nature of the two conductors in contact and upon the temperature of the junction. The value of this contact electromotive force between metallic conductors is quite small, only a small fraction of a volt (*e.g.*, between copper and zinc at 25 degrees centigrade it is 0.00045 volt), and in practical work it is therefore usually negligible. However, in case of a metal conductor in

contact with an electrolyte the contact electromotive force may be several volts. In this case, only a very small part of the electric energy absorbed by or given out by the current manifests itself as a loss or gain of heat energy at the junction of the two conductors, but the gain or loss of electric energy by the current appears as a loss or gain of *chemical* energy at the junction of the two conductors. That is, the transfer of energy involved in the chemical changes which take place at the junction is many times greater than the transfer of heat energy from or to the junction between dissimilar *metals*. As in the case of two metals in contact, however, this contact electromotive force between conductors and electrolytes is independent of the strength or the direction of the current through the junction and is also independent of the area and the shape of the surface of contact, but depends only upon the nature of the conductors in contact and the temperature of the conductors at the junction.

It is found by experiment that whenever any number of conductors are connected in series, and there are no electromotive forces in this chain of conductors other than the electromotive forces of contact at their junctions, then the net electromotive force between the ends of this series of conductors is the same as if the two end conductors were connected directly to each other, *provided all the conductors are kept at the same uniform temperature and the chemical action at the cathode in each electrolyte in the series is just the reverse of the chemical action at the anode in this electrolyte.* This fact is sometimes called the Law of Successive Contacts. For example, if a copper wire is connected to an iron wire which in turn is connected to an aluminum wire, the net electromotive force between the free ends of the copper and the aluminum wire is the same as if the copper wire were connected directly to the aluminum wire. Similarly, when two copper wires are soldered together, the net effect of the solder is nil, whether the two copper wires are in actual contact or are joined only through the solder. (If the flux used in soldering acts chemically on the wires, an electromotive force may be produced at the junction; this electromotive force, although small, may be sufficiently large to cause considerable trouble when delicate measurements are to be made.) Again, the net electromotive force of a silver voltameter is zero, since at the anode silver goes into solution as silver nitrate and at the cathode silver is deposited from

the silver nitrate solution, that is, the chemical actions at the two electrodes are just the reverse of each other.

The two exceptions to this law of successive contacts are

1. When the chemical actions at the two electrodes in any electrolyte in the series are not the reverse of each other, and

2. When the conductors forming the series are not at the same temperature.

In either case, when the conductors forming the ends of the series are connected together, there will in general be a current established in this closed chain, since the net rise of potential due to the net electromotive force in the series must be balanced by an equal fall of potential, for the total drop of potential around any closed circuit is always zero. The value of the current established in the closed circuit thus formed will be equal to the resultant or net electromotive force divided by the *total* resistance of all the conductors in the series (see Article 98).

95. Chemical Batteries. — The action of all chemical batteries is based upon the first exception to the Law of Successive Contacts. For example, in the case of the simple copper-sulphuric acid-zinc battery described at the beginning of this chapter, the electromotive force of contact between the zinc and the sulphuric acid solution is in the direction from the zinc to the acid and is about one volt greater than the contact electromotive force between the copper and the acid. This latter electromotive force is in the direction from the copper to the acid; hence the net electromotive force of the battery from the zinc pole to the copper pole is about one volt. The copper pole is the positive pole and the zinc pole the negative pole.* That is, there is a net rise of potential through the battery from the zinc to the copper pole equal in value to about one volt. Consequently, when the two poles of the battery are connected by a copper wire, a current is established in this wire equal in value to this electromotive force divided by the total resistance of *all* the conductors in series. For example, calling the electromotive force of the battery 1 volt, the resistance of the wire 1 ohm,

*The standard symbol for a battery is two parallel lines thus

The short line represents the negative and the long line the positive pole.

and the resistance of the conductors forming the battery 0.2 ohms, the current in the circuit will be $\frac{1}{1.2}=0.833$ amperes. The resistance of the conductors forming the battery is usually called the *internal resistance* of the battery, and is never negligible unless the wire forming the external circuit has a very large resistance. It should also be noted that in the example just cited there is also a contact electromotive force between the zinc pole and the copper wire forming the external circuit, but since this electromotive force is but a very small fraction of a volt it is usually negligible in practical work. For a description of the construction of the common forms of chemical batteries used in practice, see any electrical engineer's handbook.

It is found by experiment that the net electromotive force of any chemical battery or cell is *constant*, provided the chemical nature of the electrodes and the electrolyte in contact with them does not change. One electrode may waste away and the other increase in mass, due to the chemical action which takes place at them, but as long as their chemical composition and that of the electrolyte in contact with them does not change and the temperature remains constant, the electromotive force of the battery remains constant. In general, however, when an electric current is established through a battery, the electrodes not only change in mass, but some of the products of the chemical actions which take place collect at the electrodes, and thus change the nature of the substances in contact, and consequently the electromotive force of the battery changes. For example, when a current is established through the copper-sulphuric-acid-zinc battery, zinc sulphate is formed at the zinc electrode, and bubbles of hydrogen gas collect at the copper electrode and the electromotive force of the battery falls off. The battery in such a case is said to become *polarised,* and the decrease of its electromotive force is said to arise from a *back electromotive force of polarisation.* There are various methods for preventing the polarisation of a battery; see any electrical engineer's handbook. The polarisation of a dry battery is particularly noticeable, since the electrolyte in such a battery, instead of filling the entire space between the poles of the battery, simply impregnates a practically solid mass between these poles. Consequently, the products of the chemical actions at the poles of the battery cannot

diffuse rapidly through the battery but collect at the poles where they are formed. However, when such a battery is left open-circuited for a time after it is used, these products gradually diffuse through the battery, and its electromotive force gradually returns to practically its original value, provided none of the active materials has been completely destroyed.

96. Definition of the International Volt. — Many of the ordinary forms of batteries undergo chemical changes in their various parts even when left open-circuited. This is due chiefly to impurities in the chemicals of which they are made. It is possible, however, to construct a battery which will remain practically unaltered for several years, provided no current is taken from it. Hence such a battery, or cell, makes a very convenient standard of electromotive force or potential difference. One of the most satisfactory cells of this kind is that known as the Clark cell, and in terms of this cell the International Congress of Electricians defined the volt as follows:

"As a unit of electromotive force (shall be taken) the International Volt, which is the E.M.F. that steadily applied to a conductor whose resistance is one international ohm, will produce a current of one international ampere, and which is represented sufficiently well for practical use by $\frac{1000}{1434}$ of the E.M.F. between the poles or electrodes of the voltaic cell known as Clark's cell, at a temperature of 15° C., and prepared in the manner described in the accompanying specification (B)."

The specification referred to is to be found on page 10 of *Foster's Electrical Engineer's Pocket Book.* This definition has been legalized by most civilized countries.

In the practical use of a Clark cell as a standard of comparison of potential differences, an arrangement is used which obviates the necessity of taking any current from the cell. The principle of this method is to balance the electromotive force of the cell against an equal resistance drop. See Article 100.

97. Thermal Electromotive Forces. — As noted in Article 94 the Law of Successive Contacts does not hold in case the temperature of the chain of conductors forming the electric circuit is not the same for all these conductors. For example, when an iron and a copper wire are connected to each other at their two ends in such a manner that they form a closed loop, the electromotive force, and therefore the current, in this loop will not be

zero if the two junctions between the iron and the copper are kept at different temperatures, or even if there is a difference of temperature between any two points of the same wire. These *thermal electromotive forces*, as they are called, are however only a small fraction of a volt even for a considerable difference of temperature, and consequently in any circuit in which there are other electromotive forces of the order of a volt or more, they may be neglected. One important practical application of these thermal electromotive forces, however, is in the thermo-electric couple or electric pyrometer for measuring high temperatures. The ordinary form of electric pyrometer consists essentially of a platinum and a platinum-iridium wire fused together at one end and connected in series with a millivoltmeter (*i.e.*, a voltmeter designed to measure differences of potential of the order of a thousandth of a volt). The junction between the two wires, suitably protected by a porcelain or quartz tube, is placed in the furnace the temperature of which it is desired to measure. The difference of potential indicated by the voltmeter is then practically proportional to the difference between the temperature of the hot junction in the furnace and the temperature of the ends of the two wires where they are connected to the voltmeter.

98. Kirchhoff's Laws. — We have already had occasion, in several instances, to make use of the two experimental facts that

1. The algebraic sum of the currents coming up to any junction in a network of conductors is always zero, and

2. The algebraic sum of the potential drops around any closed loop in a network of conductors is always zero.

These two experimental facts are known as *Kirchhoff's Laws*, from the name of the scientist who first clearly enunciated them. By making use of these facts one can always predetermine (1) the currents in each branch of a network when the resistance of each branch and the electromotive force in each branch are known, or (2) the resistance of each branch of a network when the current in each branch and the electromotive force in each branch are known, or (3) the electromotive force in each branch when the current in each branch and the resistance of each branch are known. These two laws are therefore of fundamental importance.

It should be carefully borne in mind in applying these laws that a current *leaving* any point is equivalent to an equal *negative*

current *entering* that point, and that an *e. m. f.* in any chosen direction is equivalent to a *rise* of potential in that direction. In working out any problem concerning a network of circuits it is convenient to make a diagram of the network and to place on each branch in this diagram a number or symbol to represent the value of the current in this branch and an arrow or subscripts to indicate the direction of the current represented by this number or symbol, and wherever there is an *e. m. f.* to place a number or symbol to represent the value of this *e. m. f.* and an arrow or subscripts to indicate its direction. Then at any junction point those currents represented by arrows pointing toward the point are to be considered positive (say) and those represented by arrows pointing away from the point are to be considered negative; and for any closed loop those currents and *e. m. f.'s* represented by arrows pointing around the loop in the clockwise direction (say) are to be considered positive and those pointing around the loop in the counter-clockwise direction are to be considered negative. With this understanding, we then have,

$$\Sigma\, I = 0 \text{ at every point} \tag{19a}$$

$$\Sigma\, E = \Sigma\, R\, I \text{ for every closed loop} \tag{19b}$$

where I, R, and E represent the current, the resistance and the *e. m. f.* respectively in each branch of the loop, and the symbol Σ indicates the algebraic sum of the expression following it.

Equations (19) enable one to write down a set of simultaneous equations for the given network, but it will be found that at least one of the current equations may be derived directly from the other current equations, and that at least one of the potential equations may be derived from the other potential equations. That is, the number of independent equations of each form will be one less than the number which it is possible to write down. It should also be noted that it is frequently unnecessary to write down formally all the possible independent equations; many of the simpler problems can be solved by writing down two independent expressions for the potential drop between each pair of points and equating these two expressions; this is illustrated by the Generalized Ohm's Law (Article 93) which is but a special case of Kirchhoff's second law. The following examples will serve to indicate the use of Kirchhoff's Laws:*

*The solution of network problems by means of determinants is given in detail in Del Mar's *Electric Power Conductors*.

a. Resistances and Electromotive Forces in Series. — Consider any number of conductors in series (Fig. 58) and let a drop of potential V_{12} be established between the two ends of this series of conductors. Let the current in these conductors be in the direction from 1 to 2 and let I_{12} be its value; I_{12} will be the same in each conductor, since they are in *series*. Let R', R'', R''', etc., be the resistances of the various conductors and E'_{12}, E''_{12}, E'''_{12}, etc., the electromotive forces in this portion of the circuit between the points 1 and 2 in the direction from 1 to 2. Then the potential drop from 1 to 2 is also $R'I_{12}-E'_{12}+R''\,I_{12}-E''_{12}+R'''\,I_{12}-E'''_{12}$, etc. Hence

$$V_{12}=(R'+R''+R''', \text{ etc.})\ I_{12}-(E'_{12}+E''_{12}+E'''_{12}, \text{ etc.}). \quad (20)$$

Therefore the resistances between the points 1 and 2 are equivalent to a single resistance

$$R=R'+R''+R''', \text{ etc.} \quad (20a)$$

and the electromotive forces between the points 1 and 2 are equivalent to a single electromotive force

$$E_{12}=E'_{12}+E''_{12}+E'''_{12}, \text{ etc.} \quad (20b)$$

When the equivalent electromotive force E_{12} is positive and greater than the product of the current I_{12} and the equivalent resistance R,

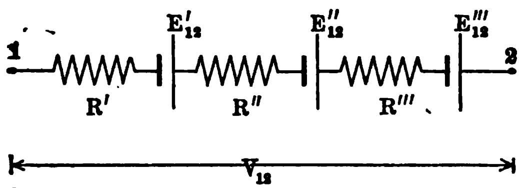


Fig. 58.

the drop of potential from 1 to 2 will be negative, that is, there will be an actual rise of potential from 1 to 2; this corresponds to the condition when the part of the circuit from 1 to 2 is supplying energy to the rest of the circuit which completes the closed loop from 2 to 1. When E_{12} is negative or is less than the product $R\,I_{12}$, the potential drop from 1 to 2 is positive, which represents a transfer of energy to this part of the circuit from the portion of the circuit closing the loop from 2 to 1.

b. Resistances and Electromotive Forces in Parallel. — When a number of resistances R', R'', R''', etc., are connected in parallel between two points 1 and 2, and there are electromotive forces

E'_{12}, E''_{12}, E'''_{12}, etc., in these respective branches, the drop of potential V_{12} from the point 1 to the point 2 must be the same for each branch, and the total current entering the point 1 must be equal to the total current leaving this point. Hence, calling I_{12} the total current entering the point 1 in the direction from 1 to 2, and I'_{12}, I''_{12}, I'''_{12}, etc., the currents in the respective branches in the direction from 1 to 2, we have

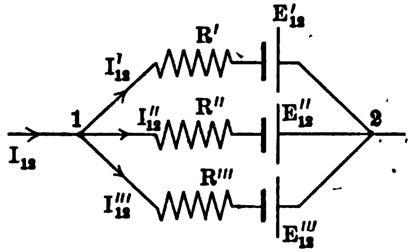


Fig. 59.

$$I_{12}=I'_{12}+I''_{12}+I'''_{12}+\text{etc.} \tag{21}$$

and

$$R'\,I'_{12}-E'_{12}=R''\,I''_{12}-E''_{12}=R'''\,I'''_{12}-E'''_{12}=\text{etc.} \tag{21a}$$

from which relations the currents in the individual branches may be calculated when the resistances and the electromotive forces in these branches are known. Again, if the drop of potential V_{12} from 1 to 2 is given instead of the total current I_{12}, we have instead of the first equation the relations

$$V_{12}=R'\,I'_{12}-E'_{12}=R''\,I''_{12}-E''_{12}=R'''\,I'''_{12}-E'''_{12} \tag{21b}$$

which are also sufficient for the calculation of the currents in the individual branches. In case there are no electromotive forces in the branches between the points 1 and 2, we have from equation (21*b*), putting $R=\frac{V_{12}}{I_{12}}$, where I_{12} is the total current from 1 to 2, that

$$R\,I_{12}=R'\,I'_{12}=R''I''_{12}=R'''\,I'''_{12}$$

whence

$$\frac{I'_{12}}{R}=\frac{I_{12}}{R'};\ \frac{I''_{12}}{R}=\frac{I_{12}}{R''};\ \frac{I'''_{12}}{R}=\frac{I_{12}}{R'''},\ \text{etc.}$$

therefore, adding these equations and substituting for I_{12} its value from equation (21), gives

$$\frac{1}{R}=\frac{1}{R'}+\frac{1}{R''}+\frac{1}{R'''}+\text{etc.} \tag{21c}$$

That is, the *equivalent resistance* R of any number of branch circuits in parallel, *in which there are no electromotive forces*, is the reciprocal of the sum of the reciprocals of the resistances of the individual branches; or, the equivalent conductance $\left(=\frac{1}{R}\right)$ of any number of branch circuits in parallel, *in which there are no electro-*

motive forces, is equal to the sum of the conductances of the individual branches. It also follows from equation (21*b*) that when there are any number of branch circuits connecting any two points, *and there are no electromotive forces in these branches,* the total current divides among the various branches in such a manner that the ratio of the current in any branch to the current in any other branch is equal to the inverse ratio of the resistances of these two branches. It should be carefully noted that none of these relations are true when there are electromotive forces in any of the branches. In the special case of two resistances in parallel, *but no e. m. f. in either branch,* equation (21*c*) becomes

$$R = \frac{R' \times R''}{R' + R''} \tag{21d}$$

99. The Wheatstone Bridge. — A special arrangement of electric circuits which is extensively used in the comparison of resistances is that known as the Wheatstone Bridge, and is shown diagrammatically in Fig. 60. B is a battery of any kind, G a galvanometer, and R_1, R_2, R_3, and R_4 are the resistances of the

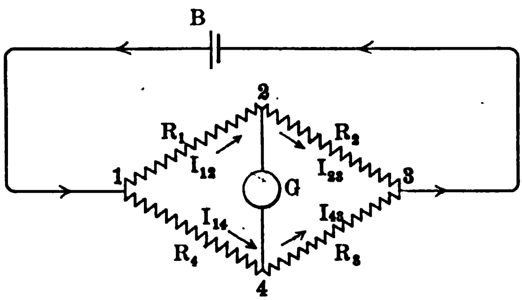


Fig. 60.

branches between the points 1 and 2, 2 and 3, 3 and 4, and 4 and 1 respectively. The currents in each branch of this network can be calculated for any values of these resistances when the electromotive force of the battery, and the resistances of the battery and the galvanometer (including the connecting leads) are known. However, there is a simple relation among the four resistances R_1, R_2, R_3, and R_4 for which the current in the galvanometer circuit will be zero, independent of the electromotive force of the battery and the resistances of the battery and galvanometer circuits. The condition that there be no current in the galvanometer is that there be no difference of potential across its terminals.

Call I_{12}, I_{23}, I_{14}, and I_{43} the currents in the branches 12, 23, 14, and 43 respectively, the order of the subscripts indicating the direction of the current in each instance. Applying Kirchhoff's first law to the points 2 and 4 we then have for no current in the galvanometer,

$$I_{12}=I_{23} \text{ and } I_{14}=I_{43}$$

and applying Kirchhoff's second law to the loops 1241 and 2342 we have

$$R_1\, I_{12}=R_4\, I_{14} \text{ and } R_2\, I_{23}=R_3\, I_{43}$$

but since $I_{12}=I_{23}$ and $I_{14}=I_{43}$, the last equation may be written $R_2 I_{12}=R_3\, I_{14}$. Whence, taking the ratio of this equation and the equation $R_1\, I_{12}=R_4\, I_{14}$ we get

$$\frac{R_2}{R_1}=\frac{R_3}{R_4}$$

or

$$R_4=\frac{R_1}{R_2}.\, R_3 \tag{22}$$

Hence, when the ratio of the two resistances R_1 and R_2 is known, and the resistance R_3 is changed until there is no current in the galvanometer, which will be indicated by the galvanometer showing no deflection when connected into the circuit, the resistance R_4 can be calculated from this equation.

In the simplest form of Wheatstone Bridge, the resistances R_1 and R_2 are formed by a continuous wire of uniform cross section and the resistance R_3 is formed of a single standard resistance coil. Instead of altering this resistance R_3, the galvanometer terminal 2 is moved along the wire until the galvanometer deflection becomes zero. The ratio of the two resistances R_1 and R_2 is then equal to the ratio of the lengths of this wire between the points 12 and 23 respectively. Calling these lengths l_{12} and l_{23} respectively, we then have that

$$R_4=\frac{l_{12}}{l_{23}}.R_3 \tag{22a}$$

Whence it is possible to determine by a very simple experiment the resistance of any conductor in terms of a single standard resistance. Hence it is necessary to measure absolutely (see Article 82) the resistance of but a single "standard" of resistance, and all other resistances can be expressed directly in terms of this standard.

100. The Potentiometer. — Another important network is that used in the so-called "potentiometer method" of comparing

potential differences. A simple arrangement of this kind is shown in Fig. 61. B is a battery or other source of electro-

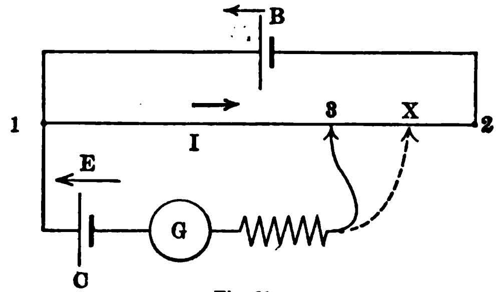


Fig. 61.

motive force which is connected to the two ends of a wire of uniform cross section. To the end 1 of this wire is also connected a standard cell C, say a Clark cell, in series with a galvanometer G and a high resistance to prevent a large current from flowing through the cell while the adjustments are being made. The other end of this circuit is an adjustable contact 3 which can be moved along the wire. The *like* poles of the two batteries must be connected to the *same* end of the wire, and the electromotive force of the battery B must be greater than the electromotive force of the standard cell. Let the contact 3 be moved along the wire until the galvanometer shows no deflection (in the final adjustment the high resistance is to be short-circuited). Then, applying Kirchhoff's second law to the loop $1CG31$, we have, since there is no current in the galvanometer, that

$$E = R_{13} I$$

where E is the electromotive force of the standard cell, R_{13} the resistance of the wire from 1 to 3, and I the current in the wire. The potential drop per unit length of the wire will then be $\frac{E}{l_{13}}$ where l_{13} is the length of the wire between 1 and 3, and the potential drop across any length of the wire, l_{1x} say, will then be $\frac{l_{1x}}{l_{13}}E$. Hence, if the standard cell is replaced by any other battery, the electromotive force E' of which is to be measured, and x is the position of the sliding contact at which the galvanometer shows no deflection, the electromotive force of this battery will be

$E' = \frac{l_{1x}}{l_{12}} E$, provided the current from the battery B, and therefore the current in the wire 12, does not change.

This potentiometer method of comparing electromotive forces is extensively employed in calibrating electrical measuring instruments. In practice, the wire 12 is usually replaced in whole or in part by a set of resistance coils, the resistances of which are accurately known.

101. Stream Lines of Electric Current. — So far we have confined our attention to conductors which are in the form of wires or long rods or bars, which we have assumed may be considered as geometrical lines. Certain problems arise in practice, however, when this method is not permissible; for example, the calculation of the insulation resistance of a cable, the calculation of the magnetic field intensity within the substance of a wire, etc. In such cases it is necessary to look upon the conductor carrying the current as made up of "current filaments," just as a magnetised body is conceived to be made up of magnetic filaments.

Experiment shows that when an insulating gap is cut in a conductor in which an electric current is established, this gap in general produces a change in the amount of the effects produced in and around the conductor, *e.g.*, a change in the force produced on the conductor by a magnetic field, a change in the amount of heating produced, etc. The amount of these changes produced depends upon the direction in which the gap is cut. It is possible to cut a very narrow gap in the conductor in such a direction that no change whatever is produced in any of these effects, just as it is possible to cut a very narrow gap in a magnet without producing any new poles (see Article 46). A conductor of any shape whatever in which a current is established may then be divided into filaments separated from each other by insulating walls of infinitesimal thickness without producing any change in the effects produced in and around the conductor. These filaments may be considered of very small cross section and may therefore be treated as geometrical lines. They may be looked upon as the lines along which the electric current flows; these filaments are called the *stream lines* of the electric current. Such stream lines are analogous to hydraulic stream lines.

Since each of these current filaments may be insulated from the rest of the conductor without altering the phenomena pro-

duced by the electric current, each filament may be treated as an insulated wire. The definitions of the strength and the direction of an electric current (Articles 66 and 67) are then directly applicable. The direction of the current at any point of a conductor is then the direction of the current filament at that point; the positive sense of the filament is taken arbitrarily as the direction of the current. The strength of the current in any filament is the same at every cross section of the filament. Let di be the current in any filament and let the cross section of this filament at any point P be ds_n, then the *current density* at P is

$$\sigma = \frac{di}{ds_n} \tag{23}$$

The total strength of the current across any surface of area S is then

$$i = \int_S (\sigma \cos \alpha)\, ds \tag{23a}$$

where ds is any element of this surface, and α is the angle between the direction of the current at ds and the normal to ds. Compare with equations (16), of Chapter II.

The results of all known experiments show that the stream lines of an electric current must be considered as closed loops without ends; in this respect these stream lines are analogous to lines of induction. In the case of a continuous current, *i.e.*, a current which does not vary with time, these stream lines are confined entirely to conductors. The stream lines of a variable current, however, may be partly or entirely in a dielectric, even though this dielectric be a perfect insulator; the stream lines in a dielectric represent the displacement current (see Chapter V).

It should be noted that the direction of the stream lines in any body depends upon the position of the points of connection of the body to the rest of the circuit. These stream lines always run through the body in the general direction of the line connecting the two points of contact or terminals. The stream lines will not in general be parallel to one another, particularly when the distance between the terminals is large compared with the linear dimensions of the cross section of the body perpendicular to the direction of these lines. However, as has already been noted, in the case of a long wire, a long rod or strip, or a long section of a rail, these lines, except for a negligible distance in

the immediate vicinity of the terminals, are practically parallel to the axis of the conductor when the conductor is of uniform cross section and the terminals are at the two ends of the conductor. Under these conditions the current density is also uniform over the cross section of the conductor provided the current does not vary with time.

In the case of a variable current the back electromotive force set up by the varying magnetic field due to the current is greater in those filaments in the interior of the conductor than in those near the surface, and as a consequence the current density is greater near the surface of the conductor; this phenomenon is known as the "skin effect." (See Article 121). In the case of copper or aluminum wires of the size ordinarily employed in practice this skin effect is not appreciable except for rapidly varying currents; in the case of rapidly alternating currents (frequency greater than 60 cycles per second) this effect may be quite appreciable. In the case of steel rails the skin effect produces a considerable increase in the apparent resistance of the rail even when frequencies as low as 15 cycles per second are employed.

102. Resistance Drop and Electric Intensity. — Electric Equipotential Surfaces. — Consider an elementary length dl of a current filament at any point P of a conductor. Let ds be the cross section of this filament at the point P. Then, from equation (11), the resistance of the elementary length dl is

$$dR=\rho\frac{dl}{ds}$$

where ρ is the specific resistance of the conducting material. The resistance drop in the length dl due to a current di in this filament is then

$$dv_r=dR.di=\rho\, dl\frac{di}{ds}$$

But $\frac{di}{ds}$ is equal to the current density σ at the point P. Hence the resistance drop *per unit length* of the filament at any point P is

$$H_e=\frac{dV_r}{dl}=\rho\,\sigma \tag{24}$$

The resistance drop per unit length of a current filament at any point is called the *electric intensity* at this point. This relation

between current density and electric intensity is of exactly the same form as the relation between magnetic field intensity and flux density (see equation 20 of Chapter II); the reciprocal of the electric resistance, *i.e.*, the specific conductivity, is exactly analogous to magnetic permeability.

The *c. g. s.* electromagnetic unit of electric intensity is the abvolt per centimeter; the practical unit is the volt per centimeter or the volt per inch. Electric intensity is also sometimes expressed as so many *c. g. s.* electrostatic units per centimeter length. These units are related to one another as follows:

1 abvolt per centimeter $=10^{-8}$ volts per centimeter
$=2.54 \times 10^{-8}$ volts per inch

1 *c. g. s.* electrostatic unit per centimeter $=300$ volts per centimeter

The resistance drop in an elementary length dl of a current filament may then be written $H_e dl$ and therefore the total resistance drop between any two points 1 and 2 on the *same* current filament is

$$V_r=\int_1^2 H_e dl \tag{25}$$

and the total resistance drop between *any* two points 1 and 2, whether on the same or different current filaments, is

$$V_r=\int_1^2 (H_e \cos\theta)\, dl \tag{25a}$$

where H_e is the electric intensity at the elementary length dl of the path between 1 and 2 and θ is the angle between the direction of the electric intensity at dl and the direction of dl. Compare with equation (25) of Chapter II.

When there is no contact or induced electromotive force in the path from 1 to 2, this resistance drop is equal to the total potential drop along this path; when there is a contact or induced electromotive force e_{12} in the path from 1 to 2 the total drop of potential from 1 to 2 is

$$V=\int_1^2 (H_e \cos\theta)\, dl - e_{12} \tag{25b}$$

Since the total electric potential drop around any closed path is zero, this reduces to

$$\int_{|L|} (H \cos\theta)\, dl=\Sigma\, e \tag{25c}$$

for a closed path; that is, the total resistance drop around any

closed path is equal to the total electromotive force in this path, which is simply another way of stating Kirchhoff's second law.

Any surface all points of which are at the same electric potential is called an *electric equipotential surface.* Such a surface is perpendicular at each point to the current filament through that point and is therefore perpendicular to the electric intensity at that point. Compare with magnetic equipotential surfaces.

103. Insulation Resistance of a Single Conductor Cable. — As an example of the use of the above conceptions, take the problem of calculating the insulation resistance of a single conductor cable in a lead sheath. Fig. 62 represents the cross section of such a cable. When all points of the wire are at the same potential and all points of the sheath are at the same potential, the surface of the wire and the inside surface of the sheath are equipotential surfaces.

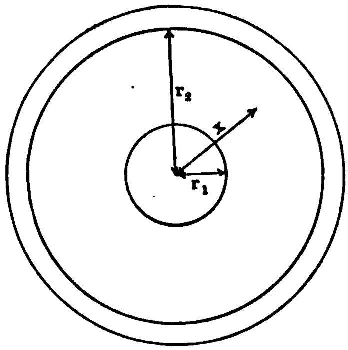


Fig. 62.

Hence the stream lines of the current through the insulation (which, it is to be remembered, is never a perfect insulator but only an extremely poor conductor) must, from symmetry, be radial lines. Let

l = length of cable in centimeters

r_1 = radius of wire in centimeters

r_2 = internal radius of sheath in centimeters

I = total current in amperes through the insulation, *i.e.*, I is the leakage current, not the main current through the wire

V = difference of potential in abvolts between the wire and the sheath, assumed constant.

ρ = the specific resistance of the insulation in ohms per centimeter-cube.

Then the current density at any point in the insulation at a distance x from the center of the wire is

$$\sigma = \frac{I}{2\,\pi\,xl}$$

whence the electric intensity at this point is

$$H = \rho\,\sigma = \frac{\rho\, I}{2\,\pi\, xl}$$

and therefore the difference of potential between the wire and the sheath is*

$$V = \int_{r_1}^{r_2} H dx = \frac{\rho\, I}{2\,\pi\, l} \int_{r_1}^{r_2} \frac{dx}{x} = \frac{\rho\, I}{2\,\pi\, l}\, ln\, \frac{r_2}{r_1}$$

The insulation resistance of the given length of cable is then

$$R = \frac{V}{I} = \frac{\rho}{2\,\pi\, l}\, ln\, \frac{r_2}{r_1} \qquad (26)$$

Note that the insulation resistance varies inversely as the length; this is evidently true when it is remembered that each elementary length of the insulation of the cable (measured along the axis of the wire) is in parallel with all the other elementary lengths.

Compare equation (26) with the formula, equation (20) of Chapter V, for the electrostatic capacity of a sheathed cable.

It can be shown that in every case the formula for the insulation *conductance* (*i.e.*, the reciprocal of the insulation resistance) between any two conductors is identical with the formula for the electrostatic capacity of these two conductors when $\frac{4\,\pi}{\rho}$ is substituted for the dielectric constant K in the formula for capacity. See Article 152 for various capacity formulas.

104. Field Intensity at any Point Due to a Current in a Wire of Circular Cross Section. — Another application of the conception of current filaments is the proof of equations (4) and (4*a*) of Article 73, for the case of a solid wire of circular cross section. A rigid proof of these equations requires the use of a geometrical theorem

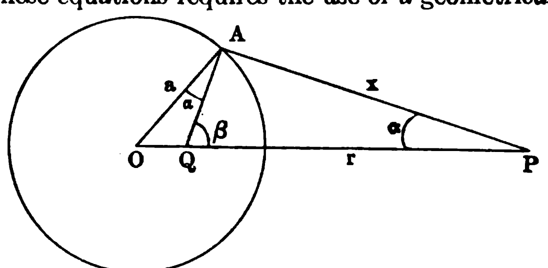


Fig. 63.

known as the Theorem of Inverse Points. This theorem may be stated thus: let P be any point either inside or outside a circle

*ln stands for the *natural* logarithm.

of radius a at a distance r from the center O of the circle, and let the point Q be on the line OP at a distance $\frac{a^2}{r}$ from the center O on the same side of O as P. Then the ratio of the distances of any point A, on the circumference of the circle, from Q and P respectively, has the constant value $\frac{a}{r}$. The two points P and Q are called *inverse points* with respect to the circle.

To prove this theorem, draw from A the line AQ, making the angle $\alpha = OPA$ with the radius OA; we wish to show that the point Q where this line cuts OP is such that $OQ = \frac{a^2}{r}$ and $\frac{QA}{x} = \frac{a}{r}$. Since the two triangles OQA and OAP have one angle in common and a second angle equal, they are similar. Hence their corresponding sides must be proportional, that is

$$\frac{OA}{OP} = \frac{OQ}{OA} = \frac{QA}{AP}$$

or

$$\frac{a}{r} = \frac{OQ}{a} = \frac{QA}{x}$$

whence $OQ = \frac{a^2}{r}$ and $\frac{QA}{x} = \frac{a}{r}$, a constant. When the point P is outside the circle, Fig. 63, its inverse point Q is inside the circle and

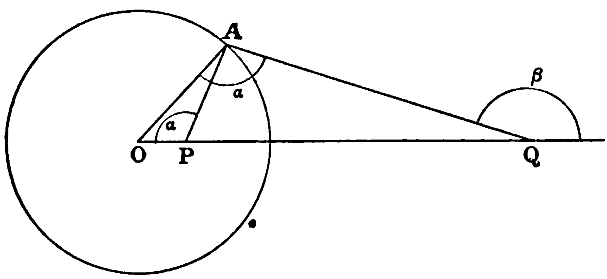


Fig. 64.

when the point P is inside the circle, Fig. 64, its inverse point Q is outside the circle. This follows immediately from the fact that $OQ = a\left(\frac{a}{r}\right)$ and therefore when r is greater than a, OQ is less than a, and when r is less than a, OQ is greater than a.

Now consider a straight hollow wire or tube of infinite length and let the walls of this tube be of infinitesimal thickness t. Let a be the radius of this tube and I the total current flowing in its walls. These walls may be considered as made up of filaments of infinitesimal cross section tds where ds is an elementary length

in the circumference of the circle representing the cross section of the tube. When the current is uniformly distributed in the walls of the tube, the current in each filament is

$$dI = \frac{tds}{2\pi at} I = \frac{I\,ds}{2\pi a}$$

Let P be any point at a perpendicular distance r from the axis of the tube, and let x be the perpendicular distance of P from a filament A. Then from equation (34*a*) this filament produces a field intensity

$$dH = \frac{2dI}{x} = \frac{I}{\pi a}\,\frac{ds}{x}$$

in the direction perpendicular to AP. (The direction of the current is assumed up toward the reader.) This intensity may be resolved into two components, one parallel to OP and the other perpendicular to OP. A second filament ds' of equal cross sec-

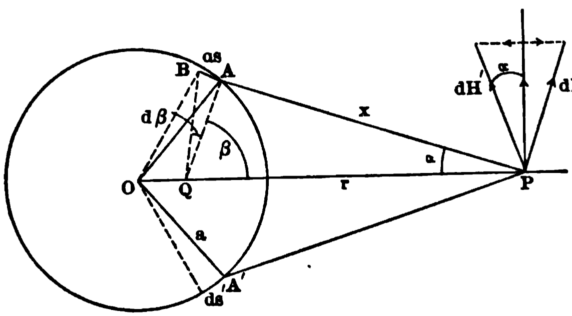


Fig. 65.

tion symmetrically located at A' on the other side of the line OP will produce an equal field intensity dH' at P in the direction perpendicular to $A'P$. Its component parallel to OP will be equal and *opposite* to the component of dH in this direction. Similarly for any other pair of symmetrically located filaments. Hence the resultant field intensity at P due to the entire tube is equal to the sum of the components perpendicular to OP of the field intensities due to all the filaments, and will be in the direction perpendicular to OP.

The component perpendicular to OP of the field intensity at P due to the filament ds is

$$dH_n = \frac{I}{\pi a}\,\frac{ds.\cos\alpha}{x}$$

Let Q be the inverse point of P with respect to the circle representing the cross section of the tube, and let β represent the angle PQA and $d\beta$ the angle at Q subtended by the arc ds. Draw a line AB through A perpendicular to QA; this line will make the angle α with ds, since the angle between OA and QA is α, and OA

is perpendicular to ds and QA to AB. Hence the projection of ds on this line AB is $(ds \cos \alpha)$. Hence, since ds is infinitesimal, $d\beta = \frac{ds \cos \alpha}{QA}$. But since Q and P are inverse points $\frac{QA}{x} = \frac{a}{r}$ whence $QA = \frac{ax}{r}$. Therefore $d\beta = \frac{rds . \cos \alpha}{ax}$, or

$$\frac{ds \cos \alpha}{x} = \frac{a}{r} d\beta$$

Hence

$$dH_n = \frac{I}{\pi a} \frac{a}{r} \; d\beta = \frac{I}{\pi r} d\beta$$

and therefore the total field intensity at P when P is *outside* the tube is

$$H = \int_{\beta=o}^{\beta=2\pi} dH_n = \frac{I}{\pi r} \int_{\beta=o}^{\beta=2\pi} d\beta = \frac{2I}{r}$$

Therefore the field intensity at any point outside a tube of circular cross section carrying a current I is the same as would be produced by a current of the same strength concentrated in a line coinciding with the center of the tube. Since a solid wire of circular cross section may be considered as made up of a series of concentric tubes, this formula is likewise true for any point P *outside* a *solid* circular wire.

When the point P is *inside* the tube the inverse point Q is outside. Hence in this case the two limiting values of the angle β are identical and equal to π; therefore the total field intensity at P is

$$\frac{I}{\pi r} \int_{\beta=\pi}^{\beta=\pi} d\beta = 0$$

That is, the field intensity at any point *inside* a circular tube in which the current is uniformly distributed is *zero*.

At any point P inside a *solid* circular wire in which the current is uniformly distributed the field intensity is therefore due only to the current *inside the cylinder the cross section of which is the circle through P concentric with the center of the wire.* Calling r the distance of the point from the center of the wire, a the radius of the wire and I the total current, the current in this cylinder is then $\frac{\pi r^2}{\pi a^2} I = \frac{r^2}{a^2} I$, since πr^2 is the area of the cross section of the cylinder and πa^2 is the area of the cross section of the whole wire. Hence the field intensity at the point P *inside* the wire is

$$H_i = \frac{2\left(\frac{r^2}{a^2} I\right)}{r} = \frac{2Ir}{a^2}$$

That is, at any point *inside* the solid circular wire the field intensity varies *directly* as the distance of the point from the center of the wire, while *outside* the wire the field intensity varies *inversely* as the distance of the point from the center of the wire.

SUMMARY OF IMPORTANT DEFINITIONS AND PRINCIPLES

1. As the measure of the **strength of an electric current** in a wire is taken the ratio of the force per unit length of the wire, which would be produced on the wire by a magnetic field, to the component of the flux density of this field perpendicular to the wire. The *c. g. s.* electromagnetic unit of current strength is the **abampere.** The practical unit is the **ampere**

1 abampere = 10 amperes.

2. **Left-hand Rule.** The direction of the current (I) in a wire is the direction in which the middle finger of the left hand points when the thumb, forefinger and middle finger of this hand are held mutually perpendicular and the thumb is pointed in the direction in which the wire tends to move and the forefinger is pointed in the direction of the component of the flux density perpendicular to the wire.

3. A **continuous electric current** is a current the strength of which does not vary with time.

4. A **conductor** is a substance in which, when connected to the poles of a battery, a continuous electric current is established. An **insulator** or **dielectric** is a substance in which, when connected to the poles of a battery, no continuous current, or only a very small continuous current, is established.

5. Two or more conductors connected end to end in such a manner that the same current flows through each are said to be connected **in series.** Two or more conductors joining any two points of an electric circuit in such a manner that the total current entering these conductors at one point leaves these conductors at the other point are said to be connected **in parallel.**

6. The **mechanical force** acting on a wire l centimeters long carrying a current of I abamperes **due to the magnetic field** in which the wire is placed is

$$F = \int_{o}^{l} I\,(B \sin \theta)\,dl \qquad \text{dynes}$$

where B is the flux density in gausses at any elementary length

dl of the wire and θ is the angle between the direction of *B* and the direction of *dl*. For a straight wire in a uniform field this reduces to

$$F = IBl \sin \theta \qquad \text{dynes.}$$

7. The **magnetic field intensity** at any point *P* **due directly to a current** of *I* abamperes in a wire *l* centimeters long is

$$H = \int_0^l \frac{(I \sin \theta)}{r^2} dl \qquad \text{gilberts per cm.}$$

where θ is the angle between *dl* and the line drawn from *P* to *dl* and *r* is the distance from *P* to *dl*. The field intensity due directly to the current is independent of the magnetic nature of the bodies in the field; if any magnetic poles are induced by this field the field intensity due to these poles must be added (vectorially) to the above.

The magnetic field intensity at a point due to a current of *I* abamperes in a **very long straight wire** of circular cross section at a point *r* centimeters from the wire is

$$H = \frac{2I}{r} \qquad \text{gilberts per cm.}$$

when the point is **outside** the wire. When the point is **inside** a wire having a radius of *a* centimeters and the wire is **solid**, the field intensity is

$$H = \frac{2Ir}{a^2} \qquad \text{gilberts per cm.}$$

provided the current density in the wire is uniform.

The field intensity at the center of a **circular coil** with **a concentrated winding** of *N* turns carrying a current of *I* abamperes is

$$H = \frac{2\pi N I}{r} \qquad \text{gilberts per cm.}$$

where *r* is the mean radius of the coil in centimeters.

8. The **lines of magnetic induction due to an electric current** are always closed loops linking the current. The positive sense of these lines is the same as the direction in which a **right-handed screw** placed along the wire must be turned to advance the screw in the direction of the current.

9. Any substance the constituents of which are separated when an electric current is established in it is called an **electrolyte** and the process of separation is called **electrolysis.** The terminal at which the current enters an electrolyte is called the **anode** and the

terminal from which the current leaves the electrolyte is called the **cathode**; the name **electrode** is used for either terminal.

10. The relation between the mass m of a substance deposited in time t by a continuous current I and this current is

$$\frac{m}{t}=kI$$

where k is a constant, called the **electrochemical equivalent** of the substance, which constant depends on the nature of the substance and the units in which the various quantities are measured. For a variable current this formula becomes

$$\frac{dm}{dt}=ki$$

The electrochemical equivalent of a substance is proportional to its chemical equivalent.

11. The **quantity of electricity** Q which flows through any section of a conductor is defined as the product of the strength I of the current in this section by the time t during which the current flows, *i.e.*,

$$Q=It$$

This applies only to a continuous current; in the case of a variable current the quantity is

$$Q=\int_0^t idt$$

When the current is expressed in abamperes and the time in seconds the unit of quantity is the **abcoulomb**; when these quantities are expressed in amperes and seconds respectively, the unit is the **coulomb**.

1 abcoulomb = 10 coulombs

12. The **electric resistance** of a conductor is defined as the ratio of the rate at which a continuous current produces heat energy in the conductor to the square of the strength of this current, provided the conductor is of uniform structure and is kept at uniform temperature throughout. The power dissipated in a conductor by a continuous current of strength I, due to the resistance R of the conductor is

$$P_h=RI^2$$

This formula gives the power in ergs per second when the current I is expressed in abamperes and the resistance R is expressed in *c. g. s.* electromagnetic units or abohms; when the current is ex-

pressed in amperes and the resistance in ohms (the practical unit) this formula gives the power in watts.

$$1 \text{ ohm} = 10^9 \text{ abohms}$$

13. The **resistance of a conductor of length** l **and cross section** A is

$$R = \rho \frac{l}{A}$$

where the factor ρ, called the **specific resistance** of the conductor, is a constant for a given material at a given temperature. This formula applies only when the stream lines of the current are parallel and perpendicular to the section A and the current density is uniform over this section. When ρ is expressed in ohms per centimeter-cube the dimensions of the conductor must be expressed in centimeters; when ρ is expressed in ohms per inch-cube the dimensions must be expressed in inches; when ρ is expressed in ohms per mil-foot the length must be expressed in feet and the cross section in circular mils.

14. The reciprocal of electric resistance is called the electric **conductance.**

15. The variation of the resistance of a conductor with temperature is expressed by the formula

$$R_t = R_o (1 + \beta t)$$

where R is the resistance at zero degrees centigrade, R the resistance at any other temperature t degrees centigrade, and β is the **temperature coefficient of resistance.** The temperature coefficient of copper is 0.0042; the temperature of other pure, non-magnetic conductors has approximately the same value. When this value of the temperature coefficient is employed the relation between the resistances R_t and $R_{t'}$ at any two temperatures t and t' degrees centigrade is given by the formula

$$R_{t'} = \frac{238 + t'}{238 + t} R_t$$

16. The **electric energy gained by the current** in any part of a circuit is defined as the amount of energy lost by this part of the circuit or by any other bodies as the result of the existence of the current in this part of the circuit; the **electric energy lost by the current** in any part of a circuit is defined as the amount of energy gained by this portion of the circuit or any other bodies as the result of the existence of the current in this part of the circuit.

17. The **drop of electric potential** from any point 1 to any point 2 along a wire carrying an electric current is defined as the ratio of the power developed by the current between these two points to the value of the current from 1 to 2. Hence the power developed by a current I (*i.e.*, the rate at which electric energy is lost by the current) in any portion of a circuit in which the drop of potential in the direction of the current is V is

$$P = VI$$

This formula gives the power in ergs per second when the current I is expressed in abamperes and the potential drop V is expressed in *c. g. s.* electromagnetic units or abvolts; when the current is expressed in amperes and the potential drop in volts (the practical unit) this formula gives the power in watts

$$1 \text{ volt} = 10^8 \text{ abvolts.}$$

18. The drop of potential in a wire due solely to the resistance of the wire, or the **resistance drop,** is

$$V_r = RI \qquad \text{volts}$$

where all quantities are in practical units.

19. An **electromotive force** is that which produces or opposes the flow of electricity, other than the opposition due to the resistance of the conductor in which it flows. As the measure of the electromotive force in any portion of the circuit is taken the rise of potential which it would produce in the direction of the current in this portion of the circuit were there no resistance drop in this portion of the circuit. When the electromotive force produces an actual rise of potential in the opposite direction to that of the current it is called a **back electromotive force.**

20. The net rise of potential E'_{12} from terminal 1 to terminal 2 of any circuit is called the **terminal** electromotive force of the circuit when the current in this portion of the circuit is from 1 to 2; when the current in this portion of the circuit is in the direction from 2 to 1 the net rise of potential E'_{12} from 1 to 2 is called the **impressed** electromotive force, due to the rest of the circuit. The relation between the electromotive force E_{12} generated in this portion of the circuit, the current I_{12} in this portion of the circuit, the terminal or impressed electromotive force E'_{12} and the resistance R of this portion of the circuit is

$$I_{12} = \frac{E_{12} - E'_{12}}{R}$$

where all quantities are expressed in practical units. This relation is known as the **Generalized Ohm's Law**.

21. In general, an electromotive force exists at the surface of contact of any two dissimilar substances, and the value of this **electromotive force of contact** depends only upon the nature of the substances in contact and upon the temperature of the junction.

22. **Contact electromotive forces** are **appreciable** only when the substances in contact act chemically upon each other or when there are two junctions in the circuit kept at different temperatures. The electromotive force of a chemical battery is due to the contact electromotive forces between the substances of which the battery is made. The electromotive force of a thermocouple is due to the difference in the electromotive forces at the hot and cold junctions.

23. **Kirchhoff's Laws.** — These so-called "laws" are the two experimental facts:

a. The algebraic sum of the currents coming up to any junction in a network of conductors is equal to zero, *i.e.*,

$$\Sigma\, I = 0$$

b. The algebraic sum of the resistance drops around any closed loop in a network of conductors is equal to the algebraic sum of the electromotive forces in this loop, *i.e.*,

$$\Sigma\, E = \Sigma\, RI$$

24. The total **resistance** of two or more conductors **in series** is equal to the sum of the resistances of these conductors. The resultant electromotive force of two or more **electromotive forces in series** is the algebraic sum of these electromotive forces.

25. The total **conductance** of two or more conductors **in parallel** is equal to the sum of the conductances of these conductors, provided there are no electromotive forces in these conductors. The resultant resistance of **two conductors in parallel,** when there are no electromotive forces in these conductors, is

$$R = \frac{R' \times R''}{R' + R''}$$

The resultant electromotive force of two or more **equal electromotive forces in parallel** is the same as each electromotive force.

26. A conductor of any shape in which an electric current is established may be divided into filaments separated from each other by insulating walls of infinitesimal thickness without altering any of the effects produced in or around the conductor, pro-

vided these filaments are drawn in the proper directions. Such filaments are called the **stream lines** of the electric current.

27. The **electric intensity** at any point in a conductor coincides in direction with the stream line through that point and is equal to the product of the specific resistance ρ of the conductor and the current density σ at that point, *i.e.*,

$$H_e = \rho\,\sigma$$

The resistance drop between any two points 1 and 2 in a conductor of any shape is

$$V_r = \int_1^2 (H_e \cos\theta)\,dl$$

where H_e is the electric intensity at the element dl of the path between 1 and 2 and θ is the angle between the electric intensity at dl and the direction of dl.

28. **Electric equipotential surfaces** and stream lines of electric current are mutually perpendicular.

29. The formula for the **insulation conductance** between any two conductors is identical with the formula for the electrostatic capacity between these two conductors when 4π divided by the specific resistance ρ is substituted for the dielectric constant K in the formula for capacity.

PROBLEMS

1. A straight wire carrying a current of 20 amperes is placed in the gap between the poles of two magnets, the pole-faces of which are each 1 inch square. If the flux in the air gap may be represented by straight lines and the flux density is 1000 gausses, calculate the force in dynes acting upon the wire, when the wire is perpendicular to the flux lines.

Ans.: 5080 dynes.

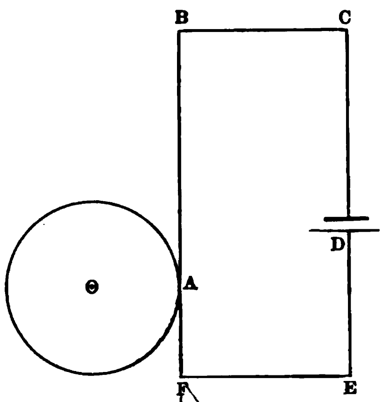


Fig. A

2. Given the electrical circuit shown in the figure. The wire at A is insulated so that all the current flowing at F must pass around the circular loop in a counter-clockwise direction before continuing to B. The current may be assumed to flow in

a geometrical line in all parts of the circuit and in the battery. Dimensions: Radius $OA = 2$ inches, $AB = 6$ inches, $BC = 4$ inches, $CE = 8$ inches, $EF = 4$ inches, and $FA = 2$ inches. If the current from B to C is 50 amperes, find the field magnetic intensity at O due to the entire circuit.

Ans.: 7.11 gilberts per cm.

3. A current of 100 amperes is established in a long iron rod the diameter of which is 1 inch. The rod has a permeability of 300 *c. g. s.* units. What is the magnetic flux density 0.25 inch from the center of the rod due to this current? What is the total number of lines of induction per foot length inside the rod?

Ans.: 2360 gausses. 91,400 maxwells.

4. The current established in a solution of copper sulphate ($CuSO_4$) is 100 amperes. Determine the weight of copper deposited on a platinum cathode in one hour. What quantity of electricity is transferred through the solution?

Ans.: 118.7 grams. 100 ampere-hours or 360,000 coulombs.

5. A copper wire two miles in length has a cross section of 0.07 square inch and has a resistance of 1.13 ohms. What is the length in feet of a wire of the same material 20,000 circular mils in cross section and having a resistance of 1.5 ohms?

Ans.: 3150 feet.

6. The resistance per mil-foot of copper is 9.59 ohms at 0° cent. What will be the resistance at this temperature of 40 grams of copper wire which has a cross section of 2500 circular mils and a specific gravity of 8.89?

Ans.: 0.0447 ohms.

7. The resistance of the armature winding of a given electric motor at 24° cent. is found to be 1.702 ohms. The armature resistance is again measured after the motor has been in service for several hours and found to be 1.980 ohms. What is the average increase in the temperature of the armature winding?

Ans.: 43° cent.

8. A 100-volt generator and a 50-volt battery are connected in series. The internal resistance of the generator is 1 ohm and the internal resistance of the battery 7 ohms; the resistance of each of the two wires connecting the battery and the generator is 1 ohm. What is the current in this circuit and the terminal voltage of the battery (1) when the electromotive forces of the battery and the generator oppose each other, and (2) when these elec-

tromotive forces are in the same direction around the circuit? The electromotive forces are to be assumed constant; actually, the electromotive force of the battery will depend upon the amount and direction of the current, due to the polarisation which takes place. (3) How much of the power developed by the engine driving the generator is converted into electric power in each case? (4) At what rate is energy transformed into chemical energy in the first case? (5) At what rate is chemical energy transformed into electric energy in the second case? (6) At what rate is energy transformed into heat energy in each case?

Ans.: (1) 5 amperes and 85 volts; (2) 15 amperes and 55 volts; (3) 500 watts and 1500 watts; (4) 250 watts; (5) 750 watts; (6) 250 watts and 2250 watts.

9. A house service consists of 5 (32 C.P.) lamps of 110 ohms resistance each, 8 (16 C.P.) lamps of 220 ohms resistance each and an electric heater of 10 ohms resistance, all connected in parallel. The voltage between the service wires at the entrance to the house is 115, and each of the two wires leading from the entrance to the load has a resistance of 0.1 ohm. What is the energy in kilowatt-hours delivered to the house during a period of 4 hours?

Ans.: 9.28 kilowatt-hours.

10. Three batteries with electromotive forces of 15, 20 and 25 volts respectively and with internal resistances of 4, 3 and 2 ohms respectively, have their positive terminals connected together and their negative terminals connected together. (1) What is the terminal electromotive force of each battery, and (2) what is the current in each?

Ans.: (1) 21.15 volts; (2) 1.538 amperes, 0.385 amperes, 1.923 amperes.

11. Fig. B represents a so-called "three wire system" which is largely used for distributing electric energy for light and power. *A* and *B* are two generators (the field windings are omitted for simplicity) with their electromotive forces acting in the same direction. *C* and *D* are the loads, which may be either lamps or motors. *a* and *b* are the out-

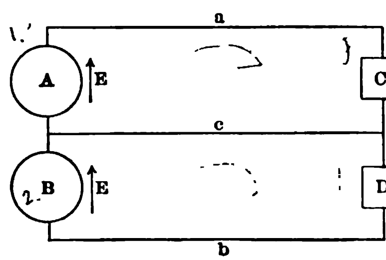


Fig. B.

side "mains" and c is the "neutral" wire. (Motors are also frequently connected across the outside mains a and b.) Let the generated electromotive forces of the two generators A and B each be 110 volts, their internal resistances be 1.5 and 2 ohms respectively, the effective resistances of the loads C and D be 8 ohms and 10 ohms respectively, and the resistance of each of the three wires a, b and c be 0.1 ohm. (1) What are the currents taken by the loads C and D? (2) What is the current in the neutral wire c? (3) What are the terminal voltages of the generators A and B, and (4) the impressed voltages at the load?

Ans.: (1) 11.43 amperes and 9.11 amperes; (2) 2.32 amperes; (3) 92.8 volts and 91.8 volts; (4) 91.4 volts and 90.8 volts.

12. Energy is delivered from a 230-volt generator to a factory over a transmission line which has a total resistance of 0.2 ohms. What is the current, the potential difference at the factory and the efficiency of transmission when the power taken by the factory is 50 kilowatts?

Ans.: There are two possible currents, depending upon the resistance of the load. These currents are 291.2 amperes and 858.8 amperes; the corresponding potential differences are 171.8 volts and 58.2 volts; the corresponding efficiencies are 74.7% and 25.3%. In practice the resistance of the load is always such that the smaller current and therefore the higher efficiency is obtained.

13. Fifty kilowatts of power are to be delivered to a factory 5000 feet from a power house. The voltage at the power house is 600. What must be the cross section of the wire in circular mils used for the transmission line in order that the efficiency of transmission be 90%? Assume the wire to have a conductivity of 98% and the temperature to be 20° cent. What would be the cost of the copper for this line if the price of copper is 20 cents per pound?

Ans.: (1) 163,800 circular mils. This corresponds approximately to a No. 000 B. & S. gauge wire. (See Appendix B.) (2) $1016 if No. 000 wire is used.

14. The inside diameter of a lead-sheathed, rubber-insulated cable which has a cross section of 250,000 circular mils is 1 inch. If the rubber has a specific resistance of 150×10^7 megohms per centimeter-cube at 20° cent., what is the insulation resistance per mile of this cable at this temperature? (Cables are usually made of stranded wires, and the cross section of the strand is therefore

not a perfect circle; in this problem, however, the wire may be assumed solid and of circular cross section.)

Ans.: 1027 megohms per mile.

15. Prove that the resistance of a uniformly tapered wire (*i.e.*, a wire such that its diameter changes by a constant amount per unit distance measured along its axis) is

$$R = \rho \frac{l}{\pi r_1 r_2} \text{ ohms}$$

where ρ is the specific resistance of the wire per centimeter-cube l is its length in centimeters, and r_1 and r_2 are the radii in centimeters of its cross section at its two ends. The current density at any cross section of the wire is to be assumed constant over that cross section. Compare with the resistance of a wire of uniform cross section.

IV

ELECTROMAGNETISM

105. Electromotive Force Due to Change in the Number of Lines of Induction Linking a Circuit.— Linkages between Flux and Current. — In the last chapter were described two types of devices for producing an electric current, the chemical battery and the thermo-electric couple. In engineering work, however, neither of these devices is used as a " generator " of electric energy, except in small amounts for testing purposes and where only small currents are required, on account of the high cost of generating electrical energy by such devices. Practically all electric generators and all electric motors are based upon an entirely different principle, namely, that *whenever the number of lines of magnetic induction linking a circuit is changed an electromotive force is produced in this circuit.* This important fact, which is the basis of electrical engineering, was discovered by Faraday in the early part of the last century. Electrical engineering as an art, however, cannot be said to date back earlier than about 1880, when the development of the incandescent lamp by Edison first created a demand for large amounts of electric energy.

The electromotive forces produced in an electric circuit by changing the number of lines of induction linking the circuit are called " induced " electromotive forces, and the currents due to these electromotive forces are called " induced " currents.

As has already been noted, every electric circuit forms one or more closed loops and every line of induction forms a closed loop. A line of induction and an electric circuit may therefore link each other in the same way that two links of a chain link each other. When the electric circuit is in the form of a coil of a number of turns the same line of induction may thread several turns of the coil; in this case the line of induction is said to form with the circuit a number of *linkages* equal to the number of turns which it threads. The total number of linkages corresponding to any number of lines of induction is the sum of the linkages of all the lines. In the special case of a coil of N turns and ϕ lines of

induction *each of which threads all these turns,* the total number of linkages is $\lambda = N\phi$; if only part of the lines thread all the turns the number of linkages will be less than $N\phi$. The symbol λ will be used throughout for number of linkages; linkages are expressed in the same unit as flux of induction, *i.e.*, in maxwells.

With this understanding the experimentally determined relation between the value of the induced electromotive force and the change in the number of lines of induction linking a circuit may be stated as follows: *The value of the electromotive force induced in an electric circuit is equal to the time rate of change of the number of linkages between the circuit and the lines of induction threading it.* Or, using the word "flux" as synonymous with "lines of magnetic induction," this law may be stated more briefly as, *the electromotive force induced in a circuit is equal to the time rate of change of the linkages between the flux and the circuit.*

The direction of this induced electromotive force is found to be such that it would of itself set up a current in the circuit in such a direction as to oppose the change in the flux linking the circuit. We have already seen (Article 72) that the direction of the lines of magnetic induction set up by a current in any circuit is always such that these lines link the circuit in the direction in which a right-handed screw placed along the wire forming this circuit would have to be turned to advance it in the direction of the current. Hence the direction of the electromotive force induced in a circuit when the flux linking this circuit is changed is opposite to the direction in which a right-handed screw placed along the conductor forming the circuit would advance if turned in the direction in which the lines of induction are *increased.* Hence, calling $d\lambda$ the *increase* in time dt of the number of linkages between the circuit and the flux threading it, the induced electromotive force in the right-handed screw direction with respect to the direction of the lines of induction linking this circuit is

$$e = -\frac{d\lambda}{dt} \tag{1}$$

Where λ is expressed in maxwells or *c. g. s.* lines of induction and t in seconds, this equation gives the value of the induced electromotive force in *abvolts.* The value of the induced electromotive force in *volts,* when λ is expressed in maxwells and t in seconds, is since 1 volt $= 10^8$ abvolts

$$e = -10^{-8} \frac{d\lambda}{dt} \tag{1a}$$

When the circuit in question consists of a single loop the linkages λ are equal to the flux ϕ, and the induced electromotive force is then

$$e = -\frac{d\phi}{dt} \text{ abvolts} = -10^{-8} \frac{d\phi}{dt} \text{ volts} \tag{1b}$$

When the circuit in question is formed by a coil of N turns of insulated wire and each line of induction links each turn $\lambda = N\phi$ and the total electromotive force induced in the coil is

$$e = -N \frac{d\phi}{dt} \text{ abvolts} = -10^{-8} N \frac{d\phi}{dt} \text{ volts} \tag{1c}$$

The minus sign in the above equations is useful to indicate the relative direction of the induced electromotive force resulting from *increasing* the lines of induction linking the circuit in the same direction as the lines of induction due to the current which may be in the circuit. The direction of the current in a circuit and the direction in which the lines of induction due to this current link the circuit are always related to each other by the right-handed screw law, while the induced electromotive force resulting from an *increase* of the number of lines of induction linking the circuit in the right-handed screw direction with respect to the current in the circuit is in the *opposite* direction to the current. In other words, *increasing* the number of lines of induction linking a circuit in the same direction as the lines of induction established by the current in the circuit, produces an electromotive force in the *opposite* direction to the direction of the current; that is, produces a *back* electromotive force. A *decrease* in the number of lines of induction in this same direction produces an electromotive force in the *same* direction as that of the current. As far as the *numerical* value of the electromotive force is concerned, the minus sign in equations (1) may be neglected.

106. Work Required to Change the Number of Lines of Induction Linking an Electric Circuit. — When the electromotive force induced in any circuit by changing the number of linkages between the flux and the circuit sets up an electric current in the circuit, or if an electric current already exists in the circuit, work must be done to produce this change in the linkages. For, if the current has the strength i and the induced electromotive force in

the direction of the current the value e at any instant, then, in any infinitesimal interval of time dt measured from this instant, the current in this part of the circuit *gains* an amount of electric energy equal to $eidt$, where e is the induced electromotive force *in the direction of the current.* But we have just seen that the value of the induced electromotive force in the direction of the current resulting from an *increase* $d\lambda$ in the number of linkages of the flux and the circuit in the same direction as the lines of induction due to this current, is $-\frac{d\lambda}{dt}$. Hence the amount of electric energy *gained* by the current when the linkages of the flux and the circuit are *increased* by an amount $d\lambda$ is

$$dW = eidt = -\frac{d\lambda}{dt}idt = -id\lambda \qquad (2)$$

When i is in abamperes and λ in maxwells this equation gives the work done in ergs.

Hence, when the number of lines of induction linking any part of a circuit is actually *increased*, the current in this part of the circuit *loses* electric energy, and when the number of lines of induction linking any part of the circuit is actually *decreased*, the current in this part of the circuit *gains* electric energy. This agrees with the general principle which has already been noted, that an electromotive force in the opposite direction to the direction of the current, that is a *back* electromotive force, corresponds to a loss of electric energy, and an electromotive force in the direction of the current corresponds to a gain of electric energy.

107. Electromotive Force Induced by the Cutting of Lines of Induction. — Right-hand Rule. — When the change in the number of lines of induction linking a wire is caused by the motion of the wire through a magnetic field, the wire may be looked upon as *cutting* the lines of induction, and the rate of change of the number of lines of induction linking the wire may be looked upon as the rate at which the wire *cuts* the lines of induction. The direction of the induced electromotive force in the wire will then be the direction in which the middle finger of the right hand points when laid along the wire and the thumb and forefinger of this hand are held perpendicular to each other and to the middle finger, with the thumb pointing in the direction of the motion of the wire and the forefinger in the direction of the flux density.

This *right-hand rule* for the direction of the induced electromotive force is equivalent to the rule given above, but is more convenient in the case of a wire moving across a magnetic field. For example, if the lines of induction are perpendicular to the plane of this page and are in the downward direction, a straight wire held parallel to the sides of the page and moved from left to right will have an electromotive force induced in it in the direction from the bottom to the top of the page, while if the wire is moved from right to left the induced electromotive force will be in the direction from the top to the bottom of the page. In the above equations for the induced electromotive force (equations 1) and the electric energy gained by the current (equation 2) in any part of a circuit, $d\lambda$ may then be interpreted either as the change in the number of linkages between the flux and the wire or as the number of lines of induction cut by the wire; in the latter case the electromotive force is to be taken as positive if its direction as determined by the right-hand rule is in the direction of the current in this part of the circuit, while if in the opposite direction to that of the current it must be taken as negative, *i.e.*, a *back* electromotive force.

When the change in the number of lines of induction linking a wire is due solely to the motion of the wire through a magnetic field, equations (1) and (2) may be deduced directly from equation (2) of Chapter III, by the application of the principle of the conservation of energy. Consider the special case of a straight wire of length l carrying an electric current i, and let this wire be in a magnetic field due to any other source. To simplify the discussion let the lines of induction be perpendicular to the wire and let the flux density have the same value at each element of the wire. In Fig. 66 the lines of induction are taken in the direction downward perpendicular to the plane of the page, the wire is taken parallel to the edge of the page and the direction of the current from the bottom to the top of the page. Then from

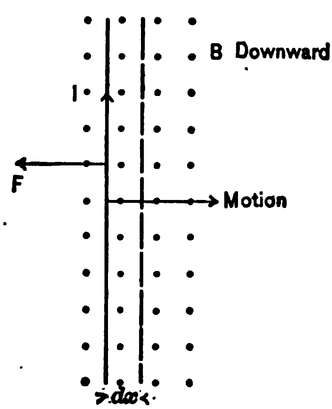


Fig. 66.

equation (2*b*) of Chapter III, the mechanical force produced on the wire by the agent producing the field is $F = Bli$, and is in the direction from the right to the left (determined by the left-hand rule, see Article 67). To move the wire a distance dx to the right then requires that some external agent do an amount of work $dW = Blidx$. But ldx is the area swept over by the wire and B is the flux density normal to this area, and since dx is but an infinitesimal distance, B may be considered constant over this area. Hence $Bldx$ is the number of lines of induction cut by the wire in moving the distance dx, that is $Bldx = d\phi$, and therefore the electric energy gained by the current is $dW = idl\,\phi$. Hence the rate at which the current gains electric energy is $i\frac{d\phi}{dt}$ and therefore the rise of potential in the direction of the current is $\frac{d\phi}{dt}$ (see equation 14, Chapter III) which by definition (Article 92) is equal to the electromotive force in this direction. The numerical values of the electromotive force and the electric energy gained by the current are identical with those given by equations (1) and (2). The direction of the induced *e. m. f.* in this case is in the direction of the current, which agrees with the *right-hand* rule stated above. If the wire is allowed to move in the direction of the mechanical force acting on it, work is done on the wire or whatever opposes its motion, and the energy for doing this work comes from a loss of electric energy by the current in the wire. In this case, the induced electromotive force is in the *opposite* direction to that of the current, which also agrees with the *right-hand* rule.

From the relation $\frac{dW}{dt} = Bli\frac{dx}{dt}$ it also follows that the value of the electromotive force in abvolts induced in the wire when it moves perpendicularly across a magnetic field may also be expressed by the formula

$$e = Blv \quad . \qquad (3)$$

where v is the velocity in centimeters per second at which the wire moves perpendicular to itself, l is the length of the wire in centimeters and B the flux density in gausses.

When the wire moves through a non-magnetic medium, the

flux density B at the wire is equal to the field intensity H, and equation (3) becomes

$$e = Hlv \tag{3a}$$

108. Intensity of the Magnetic Field inside a Long Solenoid. — A useful application of equation (2) and the conception of the cutting of lines of induction by a wire, is the calculation of the intensity of the magnetic field produced by an electric current in a coil made in the form of a long cylindrical helix of constant cross section; such a coil is called a *solenoid*. Let N' (Fig. 67) be the number of turns of wire *per centimeter length* of this solenoid, and i the current in abamperes in the wire. Let a unit north point-pole be placed at any point inside the solenoid. There will be 4π lines of *induction* radiating out from this unit point-pole, *independent* of the nature of the medium inside the solenoid, whether it be magnetic or otherwise (see Article 56). If the diameter of the solenoid is small compared to its length, practically all the 4π lines of induction which radiate out from the unit pole will pass through the lateral walls of the solenoid. Let the pole be moved a distance dx parallel to the axis of the coil; then each of these 4π lines of induction will move over a distance dx in the lateral surface of the solenoid and will therefore cut a current equal to $iN'\,dx$, provided the thickness of the insulation between successive turns may be neglected. Since there are 4π of these lines the total work done against the force produced on the unit pole by the current is $4\pi\, iN'\,dx$, provided the point at which the pole is placed is so far from the ends of the solenoid that

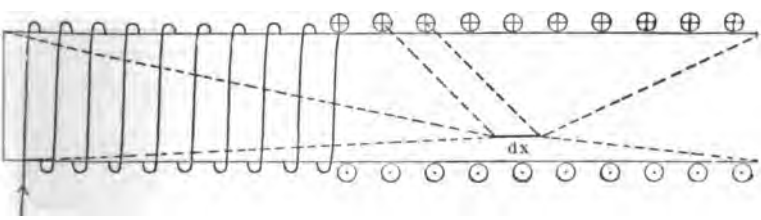


Fig. 67.

the lines which go out the ends may be neglected. Calling H the intensity of the magnetic field parallel to the axis of the solenoid due to the current in it, the work done against the force produced by the current is also equal to Hdx. Hence $H = 4\pi N'i$.

The relation between the direction of the current around the

coil and the direction of the magnetic field intensity through the coil is the same as the relation between the direction in which a right-handed screw is turned and the direction in which it advances. The reader may prove by a similar argument that the component of the field intensity at any point inside the solenoid at right angles to its axis is zero, provided the lines of induction which go out the ends may be neglected. (In proving this the relative directions of the lines of induction and the current on the two sides of the solenoid must be taken into account.) Hence the resultant field intensity in gilberts per centimeter at *any* point inside a long solenoid a considerable distance from the ends of the solenoid due *directly* to the current in the solenoid is

$$H = 4\pi N' i \tag{4}$$

and is parallel to the axis of the solenoid, where i is the current in abamperes and N' the number of turns per centimeter length. Hence the magnetic field inside a long solenoid near its center due *directly* to the current in the coil is *uniform.*

If the solenoid is wound on an iron core, magnetic poles will be induced on the ends of the core, and these poles will also produce a certain field intensity, which inside the iron will be in the opposite direction to that due directly to the current, but the field intensity due directly to the latter will be exactly the same as previously existed in the air. When a slender iron rod, of a length considerably less than the length of the solenoid, is placed inside the solenoid with its axis parallel to the axis of the solenoid, these induced poles may be assumed concentrated in points at the ends of the rod. Let the strength of these poles be m and $-m$, and the length of the rod be l centimeters. Then the field intensity at the center of the rod due to these induced poles, or the so-called " demagnetising force " due to the ends of the rod, is approximately

$$H_i = \frac{m}{\left(\frac{l}{2}\right)^2} + \frac{m}{\left(\frac{l}{2}\right)^2} = \frac{8m}{l^2}$$

and the resultant field intensity at the center of the rod is therefore

$$H_r = H - H_i = 4\pi N' i - \frac{8m}{l^2} \tag{4a}$$

An exact determination of the pole strength m is extremely difficult, and for this reason magnetic tests are seldom made on

rods; instead a ring or rod and yoke are usually employed (see Article 109).

109. Determination of the Number of Lines of Induction Linking an Electric Circuit. — Measurement of Quantity of Electricity. — From the fact that an electromotive force is induced in an electric circuit when the number of lines of induction threading the circuit is changed, it is possible by a simple experiment to measure the number of lines of induction through any region in space. Consider first a coil of N turns connected in a circuit of which the total resistance, including that of the coil, is R, and let the number of lines of induction threading this circuit be changed from ϕ_1 to ϕ_2, and let t be the time required for this change to take place. Then the average *e.m.f.* induced in the circuit during this interval is numerically $E = \frac{(\phi_1 - \phi_2)\ N}{t}$ provided each line of induction links each turn, and therefore the average value of the induced current is

$$I = \frac{E}{R} = \frac{(\phi_1 - \phi_2)\ N}{Rt}.$$

or

$$I\,t = \frac{(\phi_1 - \phi_2)\ N}{R}$$

But It is the quantity of electricity which flows through the circuit in this interval t. Hence the important relation that, when the number of lines of induction linking a coil of N turns is changed by an amount $\phi_1 - \phi_2$, a quantity of electricity

$$Q = \frac{(\phi_1 - \phi_2)\ N}{R} \tag{5}$$

is transferred across each section of the wire forming the circuit, where R is the *total* resistance of the circuit, where all quantities are in *c. g. s.* units. Hence, if when a coil which forms part of a circuit which has a *total* resistance of R abohms is pulled quickly out of a magnetic field, a quantity of Q abcoulombs of electricity is "discharged" through the circuit, the number of lines of induction in maxwells which threaded the coil when it was in the field is

$$\phi = \frac{R\ Q}{N} \tag{5a}$$

Or, if the coil is reversed in the field (*i.e.*, turned 180 degrees about

any axis in the plane of the coil), or if the direction of the field through the coil is reversed, the number of lines of induction in maxwells originally threading the coil is

$$\phi = \frac{R\,Q}{2\,N}. \tag{5b}$$

When a momentary current is established in a galvanometer of which the moving element is fairly heavy and has a long period of vibration, it can be shown that the momentary force or impulse produced on this element causes it to swing out from its position of equilibrium by an amount which is approximately proportional to the *quantity* of electricity discharged through the galvanometer.* Hence, if the coil we have just been considering is connected in series with such a long period or *ballistic* galvanometer, and the change in the number of lines of induction threading the coil is made so rapidly that the current established in the circuit lasts for only a small fraction of the time required for the moving element of the galvanometer to make a complete swing, the first swing of the moving element will be proportional (approximately) to the quantity of electricity Q discharged through the circuit, that is, proportional to $\frac{(\phi_1 - \phi_2)\,N}{R}$.

From the relation established in the preceding article, the change in the number of lines of induction, $\phi_1 - \phi_2$, can be calculated in the special case when the coil is wound on a non-magnetic spool which is placed inside and at the center of a long solenoid with an air core, and the current in the solenoid is reversed in direction. Let

N'_1 = the number of turns *per centimeter length* of the solenoid or "primary" coil.

I = the strength of the current in abamperes in the solenoid or "primary" coil.

N_2 = the number of turns on the spool or "secondary" coil.

R = the total resistance in abohms of the secondary coil, the galvanometer and any extra resistance which may be in the secondary circuit.

S = the area in square centimeters of the mean cross section of the secondary coil.

Then the field intensity at each point of the area S is $H = 4\,\pi\,N'_1\,I$,

*J. J. Thomson, *Elements of Electricity and Magnetism*, p. 377.

which is also equal to the flux density, since the permeability is unity. Hence the number of lines of induction threading the secondary coil is $4\pi N'_1 IS$. Hence, when the current in the primary coil is reversed, the change in the number of lines of induction threading the secondary coil is $8\pi N'_1 IS$. The quantity of electricity discharged through the galvanometer is then

$$Q = \frac{8\pi N'_1 ISN_2}{R} \tag{6}$$

where all quantities are in *c. g. s.* units. Hence, by calculating Q and noting the galvanometer swings when currents of various strengths are reversed in the primary coil, a curve can be plotted giving the quantity of electricity corresponding to a swing of any value; such a curve will be approximately a straight line. We have then an instrument for measuring the quantity of electricity transferred through a circuit by a momentary current, and consequently a means of determining the quantity Q in equations (4), and therefore a means for measuring the change in the number of lines of induction linking any coil.

110. Determination of the B – H Curve and Hysteresis Loop. — A useful application of equations (5) is the determination of the $B-H$ curve and hysteresis loop (see Article 57) of a sample of iron or other magnetic substance made in the form of a closed "anchor" ring or "toroid." Let such a ring be uniformly wound with a coil of wire C' which is connected through a suitable switch S and a variable resistance or rheostat R' to a battery B or other source of electromotive force. Over this primary winding on the iron ring let a second coil C be wound and connected through a resistance R'' to a ballistic galvanometer G which has been calibrated in the manner described in the preceding article. Let N_1 be the total number of turns of the primary coil, N_2 the total number of turns in the secondary coil, and R the total resistance of the secondary coil, the galvanometer and the resistance R'' in series with the coil and galvanometer. (The resistance R'' is inserted in the secondary circuit so that the galvanometer deflection may be kept within the range of the galvanometer scale.) It can be shown (Article 114) that when a current of I abamperes is established in the primary coil the lines of force set up in the ring are circles concentric with the center of the ring and the average intensity of this magnetic field is $H = \frac{4\pi N_1 I}{l}$ gilberts

per centimeter, where l is the length in centimeters of the mean circumference of the ring. (This formula is an approximation, and applies only when the radial thickness of the material forming the ring is small compared with the radius of the mean circumference of the ring, *i.e.*, the dotted line in Fig. 68.) A circular

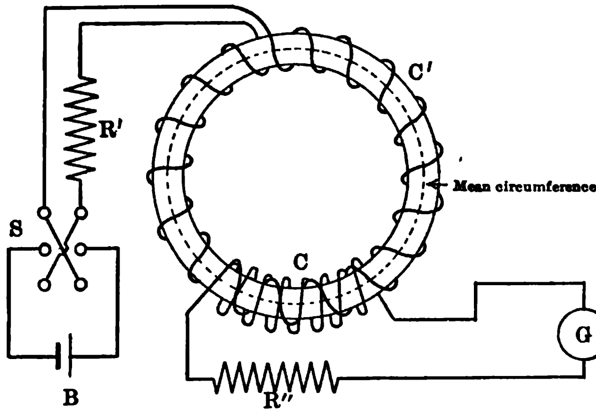


Fig. 68.

ring thus uniformly wound with a coil of wire carrying an electric current has no magnetic poles induced on it, hence this value of H is the total field intensity inside the ring and is independent of the magnetic nature of the ring. When the ring is made of a magnetic material such as iron, the flux density, *i.e.*, the number of lines of induction per square centimeter, inside the ring will not be equal to the field intensity H but will have some other value B. These lines of induction, however, will coincide in direction with the lines of force, and therefore across any radial section A of the ring there will be BA lines of induction. Hence, if the ring is originally unmagnetised* and a current of I abamperes is established in the primary winding, the number of lines of induction through the secondary coil will change by an amount $\phi = BA$ as the result of establishing in the ring a field intensity $H = \frac{4\pi N_1 I}{l}$. This change of BA lines of induction through the coil will cause a transfer of $Q = \frac{B A N_2}{R}$ abcoulombs of electricity

*The ring can be demagnetised at the start by reversing the current in the primary winding back and forth and gradually decreasing its strength.

through the galvanometer, producing a swing of the moving element. The quantity of electricity Q corresponding to this swing can be read directly from the calibration curve of the instrument, and therefore the flux density B corresponding to the field intensity H can be calculated from the formula $B = \frac{RQ}{AN_2}$.

By increasing the value of the primary current in successive steps and noting the deflection of the galvanometer corresponding to each change in the primary current, the $B - H$ curve for the sample is readily determined. To determine the hysteresis loop corresponding to any maximum field intensity H, the primary current is reversed back and forth a number of times between the positive and negative values corresponding to H and $-H$, and is then decreased in steps from the value corresponding to H to the value corresponding to $-H$, and the galvanometer deflection corresponding to each step noted. The primary current is then increased in steps from $-H$ to H, and the corresponding deflections of the galvanometer again noted. From these observations the value of the flux density corresponding to each value of H in this cycle of changes may be calculated, and hence the hysteresis loop may be plotted.

111. The Continuous Current Dynamo.— The name *continuous current dynamo* is given to any machine in which a constant or continuous electromotive force is developed by the rotation of one or more conductors in a magnetic field. Such a machine may be used as a generator of electric energy when driven by some external means such as a steam engine, water wheel or gas engine; or when electric energy is supplied to it, it may be used as an electric motor, converting electric energy into mechanical energy. The principles involved in the construction of dynamo, whether it is designed for use as a generator or as a motor, are identical.

The conductors in which the electromotive force is induced in a continuous current generator are usually insulated copper wires or copper bars, called *armature conductors*, which are imbedded in slots in the exterior surface of a hollow iron cylinder (see Fig. 69). These slots are parallel to the axis of the cylinder. The cylinder itself, which is called the *armature core*, is made up of sheets of soft iron or sheet steel which has a high permeability; the planes of these sheets are perpendicular to the axis of the

cylinder. In large machines these flat rings of sheet steel are fitted on to a cast iron frame, called the *armature spider*, which resembles the spokes and hub of a wheel. The armature spider (or, in small machines, the steel sheets themselves) is mounted on a *shaft* or axle which runs through it and projects out at each end. The ends of the shaft are mounted in suitable *bearings*, and, in case the machine is to be driven by, or is to drive, a belt, a pulley is mounted on one end of the shaft. The armature conductors, core and spider taken collectively are called the *armature* of the machine.

The magnetic field in which the armature conductors are rotated is produced by an electric current in two or more coils

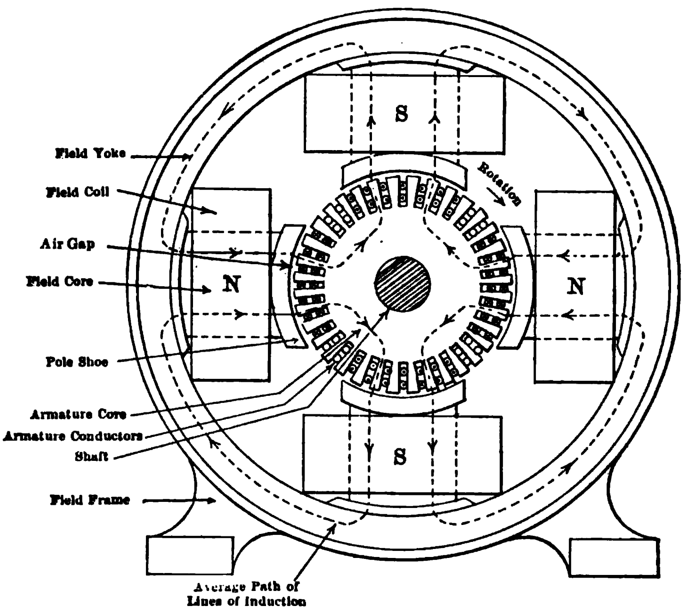


Fig. 69.

of insulated wire, *called field coils*, which are wound on stationary iron or steel cores which are placed symmetrically around the armature core, as shown in the figure. The ends of these *field cores* next the armature are broadened out so that they cover from 50 to 70 per cent of the armature surface, and are made concave toward the armature so that the end surfaces, called the *pole faces*, form part of a cylindrical surface concentric with

the armature and of a slightly greater radius. These broadened ends of the field cores are called the *pole shoes*, and are frequently made separate from the field cores and are bolted to the latter when the machine is assembled. The air space between the pole shoes and the armature core is called the *air gap*. The ends of the field cores away from the armature are connected by a yoke of iron or steel; this yoke is called the *field yoke*. The field coils, field cores, pole shoes and field yoke taken collectively are called the *field* of the machine. The field cores, pole shoes, air gap, field yoke, and armature core taken collectively are called the *magnetic circuit* of the machine, since the lines of induction are practically all confined to the space occupied by these parts. The average permeability of this magnetic circuit is high, and consequently a comparatively small amount of energy is required to maintain an electric current of sufficient strength in the field coils to establish a large number of lines of induction through the armature winding. (See Article 113.) The field cores, pole shoes and field yoke form a powerful *electromagnet* when an electric current is established in the field coils; the lines of force enter the air gap from the north pole of this magnet, and pass from the air gap into the south pole of this magnet.

In the simplest type of armature winding, the armature conductors are all connected in series by insulated copper wires or bars, called the *end connectors*, across the two ends of the armature core, and form a *closed* coil around this core. Hence the net electromotive force induced in this closed winding is the sum of the electromotive forces induced in the individual armature conductors. The field winding produces practically no lines of induction which cut the end connectors; the entire electromotive force induced in the armature winding is that due to the cutting of lines of induction by the armature conductors which lie in the air gap. Hence, when the armature conductors are distributed symmetrically around the armature core, the net electromotive force induced in the armature winding is zero, since for each conductor cutting the magnetic field on one side of the air gap, there is a corresponding conductor cutting an equal field on the other side of the armature in the opposite direction with respect to the motion of the conductor. A study of Fig. 69 will make this clear; a dot in an armature conductor indicates an electromotive force in the direction toward the reader, and a cross an

electromotive force in the direction away from the reader; the *e. m. f.* induced in those conductors not under a field pole is practically zero. The average paths of the lines of induction are shown by the dotted lines.

In spite of the fact that the net electromotive force induced in the closed armature winding is zero, a continuous electromotive force can be obtained from this machine by bringing out suitable connections or *taps* from the winding, and making contact with these connections in a suitable manner. This can be best understood from Fig. 70, which shows diagrammatically an armature winding of a simple two-pole machine, with the end connectors on the end of the armature facing the reader in heavy lines and the end connectors on the other end as dotted lines. The middle point of each of the front connections is connected to a bar of a device called a *commutator*, which is a cylinder of copper bars insulated from each other by sheets of mica. This cylinder is mounted rigidly on the shaft of the armature, from which the bars are also insulated. From the segment marked 1 there are two paths through the armature winding to the segment marked 5, and the electromotive force induced in each of the armature conductors in each of these paths is in the direction from 1 to 5. Hence the net electromotive force between 1 and 5 through each of these paths is the arithmetical sum of the electromotive forces induced in all the conductors in each path, *i.e.*, in half the total number of armature conductors. The two halves of the armature winding

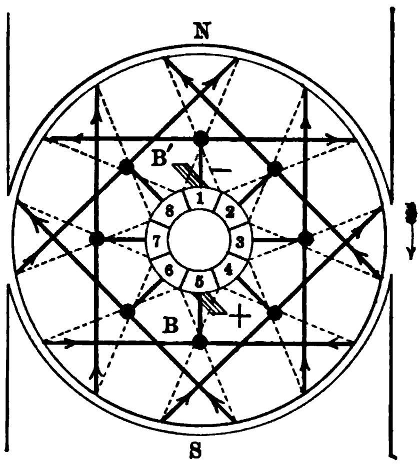
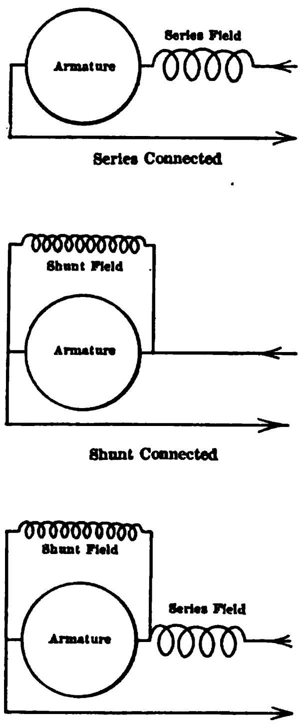


Fig. 70.

are then similar to two electric batteries in parallel; the electromotive force of one half the winding opposes the electromotive force induced in the other half of the winding and consequently no current flows in the closed circuit formed by the entire winding. However, when the segments 1 and 5 are connected by an external conductor, the electromotive force impressed on this circuit will be

equal to the electromotive force induced in each half of the winding, and consequently a current will be established in the external conductor, one half of the current flowing through each half of the armature winding; just as in the case of two batteries in parallel and connected to an external circuit half the current flows through each battery, provided the electromotive forces and the internal resistances of the two batteries are respectively equal. As the armature turns in the direction indicated the electromotive forces induced in the individual conductors forming each path between 1 and 5 will not all be in the same direction, but some of the electromotive forces will oppose the other, and hence the net electromotive force between 1 and 5 will decrease. However, when the armature has rotated through an angle corresponding to one segment of the commutator, another pair of segments, 8 and 4, come into the position formerly occupied by 1 and 5 and the electromotive force between 8 and 4 will be exactly the same as the electromotive force which previously existed between 1 and 5, provided the armature rotates with a constant speed. Similarly for the next pair of segments, and so on. Consequently, if in the positions B and B' are mounted two fixed contacts, under which the commutator segments slide as the armature rotates, the electromotive force between these two contacts will remain practically constant, and the greater the number of commutator segments the more nearly constant will this electromotive force be. The fixed contacts which rub against the commutator segments are called *brushes*, and in most modern machines are made of carbon blocks. These brushes are held in a suitable support, called the *brush-holder*, and the parts of this brush-holder in contact with the brushes are insulated by suitable bushings from the brackets which connect them to a common support called the *rocker-arm*, which is mounted on some part of the stationary structure which forms the frame of the machine. This rocker-arm is so mounted that the position of the brushes can be adjusted until no *sparking* occurs between them and the commutator segments when a current is established through the machine. The brush at the lower electric potential is called the *negative* brush, and the brush at the higher electric potential is called the *positive* brush. The electromotive force induced in a dynamo is *always* from the negative to the positive brush, whether the dynamo be used as a generator or as a motor.

When the machine is to be used as a generator it is driven by some form of "prime mover," *i.e.*, a steam engine, gas engine, or water wheel, and the circuit which is to be supplied with electric energy is connected in series with the two brushes B and B'. The field coils may be connected either in series with this external circuit; or they may be connected directly across the brushes of the machine; or there may be two sets of field coils, one set connected across the brushes and the other set in series with the external circuit. In the first case the machine is called a *series* connected generator, in the second a *shunt* connected generator, and in the third case a *compound* connected generator. These various forms of connections are shown in Fig. 71. All these types of generators are called *self-excited* generators, since they produce their own magnetic field. There is in general sufficient "residual magnetism" in the iron part of the machine to establish a weak magnetic field in the air gap, which in turn establishes a small electromotive force between the brushes when the armature is rotated; this electromotive force in turn establishes a small current in the field windings which increases the magnetic field in the gap; this in turn increases the induced electromotive force, and this cumulative process goes on until the field current reaches a steady value. In the simple shunt connected generator, for example, the field current increases until the induced electromotive force in the armature establishes a difference of potential between the brushes of the machine equal to the product of the resistance of the shunt field by the strength of the current in this field.



Compound Connected
Fig. 71.

The total current taken from the brushes of any type of continuous current generator for a given induced, or "armature," electromotive force, will depend upon the resist-

ances and electromotive forces in the external circuit and also upon the resistances of the armature circuit between the brushes and upon the resistances of the field windings. When the various resistances and electromotive forces are known the strengths of the currents in the external circuit and the various windings of the machine can be calculated by Kirchhoff's Laws. Since in an electric generator the current always gains electric energy, the direction of the current through the armature is the same as that of the electromotive force developed in the armature. Hence the current always leaves the armature of a generator at the positive brush and enters at the negative brush.

A continuous current dynamo, series, shunt or compound connected, may also be used as a motor. In this case its terminals are connected to some source of potential difference which establishes a current through its field coils and through its armature. The magnetic field produced by the current in the field coils exerts a mechanical force on the armature and causes it to rotate. Work is then done on whatever is connected to the armature, and the energy to do this work comes from the source of electromotive force, *e.g.*, the generator, which establishes the current through the machine. Since to transfer the energy from the generator to the motor it is only necessary to have two wires or *mains*, as they are usually called, the motor may be at a great distance from the generator; hence the great advantage of transmitting energy by means of an electric current. Since in an electric motor electric energy is always lost by the current in the motor, the electromotive force developed by a motor is always in the direction opposite to that of the current; that is, an electric motor always develops a back electromotive force. The current therefore always enters the armature of a motor at the positive brush and leaves it at the negative brush.

For a fuller description of the construction of electric dynamos and a discussion of the various factors which affect their operation as generators or motors, the reader is referred to any text-book on dynamo-electric machinery. In particular, it should be noted that dynamos designed for the generation or utilisation of large amounts of electric power in general have a number of pairs of poles and in most cases a corresponding number of sets of brushes, with all the positive brushes interconnected and all the negative brushes interconnected. The armature in such "multi-

polar " machines may have a single winding of the form described above, or there may be two or more independent windings on the same armature core. The general principles involved in the construction and operation of such machines are, however, identically the same as in the case of the simple two-pole machine.

112. Calculation of the Electromotive Force Induced in the Armature of a Continuous Current Dynamo. — The value of the electromotive force induced in the armature winding of a continuous current dynamo may be readily calculated from equation (1). Let

N = the total number of armature conductors.

p = the number of field poles.

p' = the number of parallel conducting paths between the positive and negative brush sets, that is, $\frac{N}{p'}$ is equal to the number of armature conductors in series between the positive and negative brush sets.

ϕ = the total useful magnetic flux per pole in *c. g. s.* lines or maxwells; that is, ϕ is the total number of lines of induction which pass through the armature core from a north pole to the adjacent south poles of the field magnet.

n = the number of revolutions of the armature per second.

The time required for each conductor to pass entirely around the armature is then $\frac{1}{n}$ and therefore the time taken for it to pass through the magnetic field under each pole is $\frac{1}{np}$. Hence the average value of the electromotive force induced in each armature conductor as it passes under each pole is $np\,\phi$ abvolts. Since the conductors are uniformly distributed around the surface of the armature, this is also the average value at each instant of the electromotive force induced in these conductors. Since there are $\frac{N}{p'}$ conductors in series between the positive and negative brush sets, the average value of the *total* electromotive force between the brushes is

$$E = \frac{np\,\phi\,N}{p' \times 10^8} \text{ volts} \tag{7}$$

When the number of commutator segments is large, the in-

stantaneous values of the electromotive force between the brushes are practically equal to this average value. For, since the brushes always make contact with segments in the *same position with respect to the field* which produces the flux, the only possible variation in the electromotive force between the brushes (when ϕ and n remain constant) is the variation that might occur as the armature rotates through an angle corresponding to one commutator segment. But in the distance corresponding to this small displacement of an armature conductor, the flux density remains practically constant (except for the three or four conductors which are just passing under or are just leaving a pole tip), and hence the *rate* at which the conductors are cutting lines of induction, and therefore the induced electromotive force, does not change appreciably in this interval.

Equation (7) holds whether the dynamo is used as a generator or as a motor. In the case of a generator, this electromotive force is in the direction of the current, and in case of a motor in the opposite direction from that of the current. It should be noted, however, that the flux per pole in a generator or motor in general depends not only upon the current in the field coils but also upon the current in the armature; this is caused by the fact that the armature current also sets up lines of induction in the opposite direction to those due to the field current and the resultant flux per pole is decreased. See under Armature Reaction in any text-book on dynamo-electric machinery.

113. Magnetomotive Force. — In equation (7) for the electromotive force of a continuous current dynamo, the only quantity which cannot be readily predetermined is the flux per pole. This quantity can, however, be calculated when the dimensions of the magnetic circuit, the permeability of its various parts, the number of turns on the field coils and the "back ampere-turns" of the armature are known. The following considerations will make this clear.

Consider a wire wound into a coil of any form having N turns, and let a current of I abamperes be established in this wire. We have already seen that a magnetic field will be established around such a coil, and that the lines of force representing this field will be closed loops linking the coil. Consequently, between any two points in the field around this coil there will be in general a drop (or rise) of magnetic potential. We wish to find the

relation between the total drop of potential around any closed path linking this coil and the value of the current established in it. By definition (Article 60), the drop of magnetic potential along any path in a magnetic field is the work done by the agent producing the field when a unit north point-pole moves around this path, that is, the drop of potential around any closed path is the line integral $\int_{|L|} (H \cos \theta)\, dl$ around that path, where dl is any elementary length in the path, H the field intensity at dl, θ the angle between the direction of dl and the direction of H, and the symbol $\int_{|L|}$ represents the integral around this closed path.

We have also seen that it is physically impossible to have a north magnetic pole without at the same time having an equal south magnetic pole on the *same* piece of matter. Hence it is physically impossible to move a unit north pole around a closed path linking a coil without at the same time linking the coil with an equal south pole, *unless the magnet at the ends of which these poles exist is threaded through and bent into a closed loop linking the coil.* Hence to move a unit north point-pole around a closed coil without at the same time having the current in the coil exert a force on the south pole connected to this north pole, we may conceive of this unit north point-pole as at the end of a flexible magnetic filament, the south pole of which is so far removed from the coil that the force exerted by the latter on this south pole is negligible. The north pole end of this filament can then be threaded through the coil and back around to its original position over the desired path, as shown in Fig. 72. The only force exerted by the coil on the filament as it is thus bent around will be the force produced by the coil on the north pole of the filament, and therefore the work done by the current will be $\int_{|L|}(H \cos \theta) dl$. But we have also seen, Article 56, that 4π lines of induction exist in a filament which has unit poles, in the direction from the south to the north pole of the fila-

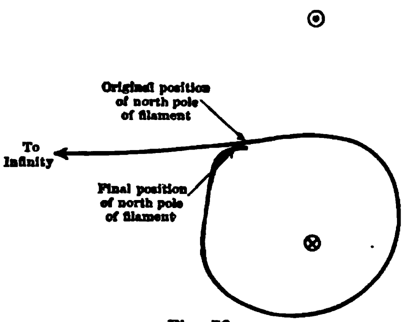


Fig. 72.

ment. Hence, when the unit north pole is moved through the coil in the right-handed screw direction with respect to the current and back again along a closed path linking the coil, the number of lines of induction through the coil is increased by 4π. Hence from equation (2), the work done by the current *on* the pole is also equal to $4\pi NI$, since the total number of linkages is $4\pi N$. Hence *the drop of magnetic potential around a closed path linking a coil of N turns in the right-handed screw direction with respect to the current in the coil is equal to $4\pi N$ times the strength of the current in the wire forming the coil.* That is

$$\int_{|L|} (H \cos \theta)\, dl = 4\pi NI \tag{8}$$

This relation is analogous to that between the resistance drop in an electric circuit and the electromotive force in the circuit, see equation (25c) of Chapter III. Hence the expression $4\pi NI$ is called the *magnetomotive force* of the coil.

Magnetomotive force is measured in the same unit as drop of magnetic potential, that is, the *c. g. s.* electromagnetic unit of magnetomotive force is the *gilbert.* The magnetomotive force is proportional to the product of the number of amperes in each turn of the coil and to the number of turns in the coil; this product NI, when I is expressed in amperes, is called the *ampere-turns* of the coil. The magnetomotive force may then be expressed as so many ampere-turns. The ampere-turn is the unit of magnetomotive force used in practice; the relation between the gilbert and ampere-turn is

1 gilbert = 0.79578 ampere-turn.

114. Magnetic Reluctance. — The analogy between the relation connecting the drop of magnetic potential and magnetomotive force and the relation between resistance drop and electromotive force may be extended still further. Consider first the simple case of a coil of insulated wire uniformly wound around a closed anchor ring, such as described in Article 110. Let the cross section of this ring be A, and the length of the mean circumference of the ring be l. From symmetry, the lines of force produced by an electric current in the coil wound on this ring will be circles concentric with the center of the ring, and each line of force must therefore link all the turns of the coil. The field intensity will also have the same value for every point on any one of these circles. Hence, calling H the field intensity at any point on the

mean circumference of the ring, the drop of magnetic potential around this circumference is

$$Hl = 4\pi NI$$

When the radius of the ring is large compared with the radius of the section A, the field intensity at the mean circumference may be taken as the average field intensity over this area A. Hence, calling μ the permeability of the iron forming the ring, the average flux density over the area A is μH, and therefore the total flux or number of lines of induction through A is $\phi = \mu HA$. Hence

$$\phi = \frac{4\pi NI}{\frac{l}{\mu A}} \tag{9}$$

The expression $\frac{l}{\mu A}$ is analogous to the expression for the resistance of a wire of length l, cross section A and specific resistance $\frac{1}{\mu}$, and it is therefore called the *reluctance* of the magnetic circuit formed by the ring.

The reluctance of any portion of a magnetic circuit has no meaning unless the two ends of the given portion of the circuit where the lines of induction enter and leave it are magnetic equipotential surfaces and the total number of lines of induction through each cross section of the given portion of the circuit is the same. When these conditions are satisfied, the reluctance R may be defined as the ratio of the difference of magnetic potential V_m between the two end faces of the given portion of the circuit to the total flux of induction ϕ through the circuit, *i.e.*,

$$R = \frac{V_m}{\phi} \tag{10}$$

When the difference of magnetic potential is expressed in gilberts and the flux in maxwells, the unit of reluctance is called the *oersted*. The relation expressed by equation (10) is of exactly the same form as Ohm's Law for an electric circuit; it is therefore sometimes called " Ohm's Law for a magnetic circuit." Magnetic flux, magnetomotive force and magnetic reluctance are strictly analogous to electric current, electromotive force and electric resistance respectively, *except that no energy is required to maintain a magnetic flux through a reluctance*, while energy is always required to maintain an electric current through a resistance.

Since lines of magnetic induction are always closed loops, the flux of magnetic induction coming up to any surface must always equal the flux of induction leaving that surface. Therefore at any junction in a network of magnetic circuits

$$\Sigma \phi = o \qquad (11)$$

This is the same as Kirchhoff's first law for an electric circuit. Similarly, since the drop of magnetic potential V_m around any closed loop is equal to the total magnetomotive force in this path,

$$\Sigma \phi R = \Sigma 4 \pi N I \qquad (11a)$$

where R is the reluctance of any closed tube of induction and ϕ is the flux through this tube. This last relation is the same as Kirchhoff's second law for a network of electric circuits.

The difficulty in applying these laws to a magnetic circuit arises from the fact that the magnetic flux is not confined to approximately geometrical lines like the currents in a network of insulated wires, but in general fills all space surrounding the coils which establish the magnetomotive forces; also, when there is iron in the circuit the permeability depends on the flux density and the previous history of the iron. (The distribution of magnetic flux in and around an iron circuit is analogous to the distribution of current in and around an uninsulated mass of copper of the same shape as the iron circuit immersed in a liquid having a conductivity about equal to that of carbon.) Ohm's Law for the magnetic circuit, however, tells us that in order to obtain a large flux with the least number of ampere-turns it is necessary to provide a path of low reluctance for the lines of induction. Hence in nearly all electric machinery a closed or nearly closed iron or steel circuit is provided for the lines of induction, since iron and steel have a high permeability, and therefore for the same dimensions a much less reluctance than a non-magnetic substance. Only in the special case of the uniformly wound circular ring discussed above, however, are the lines of induction confined *entirely* to an iron circuit; in general a certain number also exist in the air and in whatever other substances are in the vicinity of the iron circuit. For example, in the case of a dynamo, a certain percentage of the total number of lines of induction established by the field coils "leak" around through the air from one pole to the next without linking the armature conductors. The predetermination of the ratio of the total flux to the useful

flux, *i.e.*, the ratio of the total number of lines of induction to the number which link the armature conductors, can be made only very roughly; this ratio, which is called the "leakage factor," may, however, be determined by experiment; in modern dynamos is found to vary from 1.1 to 1.5, depending upon the arrangement of the field magnets.

115. Calculation of Ampere-Turns Required to Establish a Given Flux. — The following example will illustrate the way in which the number of ampere-turns required to establish a given flux may be calculated to a rough degree of approximation. Let it be required to find the number of ampere-turns necessary to establish a total flux ϕ through the armature of a two-pole dynamo. Let the armature be made of sheet steel punchings and the field cores, pole shoes, and field yoke be a single piece of cast iron.

Let

A_g = area of the air gap under each pole.

A_a = the mean cross section of the path of the lines of induction through the armature.

A_f = the mean cross section of the path of the lines of induction through the field.

l_g = the radial depth of the air gap; usually called the *length* of the air gap.

l_a = the mean length of the path of the lines of induction through the armature.

l_f = the mean length of the path of the lines of induction through the field.

k = the leakage factor.

Then the flux density in the air gap is $\frac{\phi}{A_g}$ and this is also equal to the field intensity in the air gap, that is $H_g = \frac{\phi}{A_g}$. Hence the drop of magnetic potential across both air gaps is $2 H_g l_g = \frac{2 \phi l_g}{A_g}$. The flux density in the armature is $\frac{\phi}{A_a}$; from the $B-H$ curve for sheet steel punchings find the corresponding value of H; call this value H_a. Then the drop of magnetic potential through the armature is $H_a l_a$. The total flux through the field will be

$k\phi$, and therefore the flux density in the field will be $\frac{k\phi}{A_f}$; from the $B-H$ curve for cast iron find the corresponding value of H; call this value H_f. Then the drop of magnetic potential through the field will be $H_f\, l_f$. Hence, equating the total drop of magnetic potential around the entire magnetic circuit to the total magnetomotive force $4\pi NI$ linked by the mean path of the flux, we have

$$\frac{2\,\phi\, l_g}{A_g} + H_a\, l_a + H_f\, l_f = 4\,\pi\, NI \qquad (12)$$

all in *c. g. s.* electromagnetic units. From this relation the number of ampere-turns can be calculated. For calculations of this sort it is much more convenient to have the $B-H$ curves plotted in *c. g. s.* lines per square inch as ordinates against field intensity in ampere-turns per inch. When ϕ is expressed in maxwells, H is expressed in ampere-turns per inch, l in inches, A in square inches and I in amperes, equation (8) becomes

$$\frac{\phi\, l_g}{1.596 A_g} + H_a\, l_a + H_f\, l_f = NI \qquad (12a)$$

It should be noted that the ampere-turns thus calculated are the *net* ampere-turns required to establish the given flux through the armature, that is, the difference between the ampere-turns which must be on the field coils and the back ampere-turns of the armature. (See under Armature Reaction in any text-book on dynamo-electric machinery.)

116. Self and Mutual Induction. — When the current in any electric circuit varies with time, the magnetic field produced by this current also varies with time; hence any electric circuit which is linked by the lines of induction due this varying current will have an electromotive force induced in it. In particular, the circuit in which the current in question is flowing will have an electromotive force induced in it; the electromotive force induced in a circuit due to the variation of the current in this circuit is called the electromotive force of *self induction* in this circuit. Similarly, the electromotive force induced in any other circuit 2 as the result of the variation of the current in any circuit 1 is called the electromotive force of *mutual induction* in circuit 1 due to circuit 2. If a varying current in any circuit 1 produces an electromotive force in some other circuit 2, then a varying current in 2 will likewise produce an electromotive force in 1; hence the name "mutual" for such electromotive forces.

The numerical value of the ratio of the electromotive force induced in a circuit, due to the change of the current in this circuit, to the time rate of this change, is called the *coefficient of self induction* or the *self inductance* of this circuit. That is, when the current in the given circuit varies at the rate $\frac{di}{dt}$ and, as a result of this variation, an electromotive force e is induced in this circuit, then the self inductance of this circuit is

$$L=\frac{e}{\frac{di}{dt}} \tag{13}$$

When the electromotive force is expressed in abvolts, the current in abamperes and time in seconds, the unit of self inductance is called the *abhenry;* when these quantities are expressed in volts, amperes and seconds respectively, the unit of self inductance is called the *henry*. A *millihenry* is the one-thousandth part of a henry. These units are therefore related as follows:

1 henry $=10^9$ abhenries
1 millihenry $=10^6$ abhenries

Similarly, the numerical value of the ratio of the electromotive force e_{12} induced in any circuit 1, due to a change in the current i_2 in any other circuit 2, to the time rate of the change of this current i_2, is called the *coefficient of mutual induction* or the *mutual inductance* M_{12} of circuit 2 with respect to circuit 1; that is

$$M_{12}=\frac{e_{12}}{\frac{di_2}{dt}} \tag{14}$$

Mutual inductance is expressed in the same unit as self inductance.

The self inductance of a circuit may also be expressed in terms of the number of linkages (see Article 105) between the circuit and the flux produced by the current in it. From the fundamental law of electromagnetic induction, a change in the number of linkages between a circuit and the flux linking it induces in this circuit an electromotive force equal to the time rate of change of these linkages. Hence, calling $d\lambda$ the change in the number of linkages between the circuit and the flux due to a change in the current in the circuit by an amount di, we have as another expression for the electromotive force of self induction

$$e = \frac{d\lambda}{dt}$$

Equating this value of the self-induced electromotive force to that given by equation (13) we have that the self inductance in abhenries is

$$L = \frac{d\lambda}{di} \tag{15}$$

where λ is expressed in maxwells and i in abamperes. Hence the self inductance of a circuit is equal to the change of the linkages between the circuit and the flux threading it per unit change in the current in this circuit. As will be proved presently, when the permeability of every body in the magnetic field is constant and the circuit remains unaltered in size and shape, the linkages between the circuit and the flux threading it due to the current in this circuit are directly proportional to the strength of the current in the circuit; under these conditions, therefore, the self inductance is a *constant of the circuit* (for a given distribution of current) *equal to the number of linkages between this circuit and the flux produced by unit current in it.*

Similarly, the mutual inductance in abhenries of a circuit 2 with respect to any other circuit 1 is equal to the change of the linkages λ_{12} of circuit 1 by the flux due to the current i_2 in 2 per unit change in the current in 2, *i.e.*,

$$M_{12} = \frac{d\lambda_{12}}{di_2} \tag{16}$$

where λ_{12} is in maxwells and i_2 in abamperes. When the permeability of every body in the magnetic field is constant and the two circuits are fixed with respect to each other and remain unaltered in size and shape, the mutual inductance of one circuit with respect to another is *constant* (for a given distribution of the currents) *and equal to the number of linkages between one circuit and the flux produced by unit circuit in the other;* moreover, as will be proved presently, the mutual inductance of one circuit with respect to the other is equal to the mutual inductance of the second circuit with respect to the first.

117. Proof of the Relation between Inductance and Linkages.—The intensity at any point of the magnetic field due to an electric current is given by the equation (see Article 71) $H = i\int_{|L|} \frac{(\sin\theta)\,dl}{r^2}$,

where i is the strength of the current in the circuit, dl any elementary length of the circuit, r the distance of the point in question from the elementary length dl and θ the angle between the line drawn from the point to dl and the direction of dl, and the symbol $\overline{\int}_{|L|}$ represents the vector integral of the expression $\frac{(\sin\theta)\,dl}{r^2}$ around the entire circuit. As long as the shape and position of the circuit remain unaltered, the quantity under the integral sign remains unaltered; hence the field intensity at any point due directly to a current in a given circuit is proportional to the strength of this current, no matter what the shape or size of the circuit may be, provided the shape and size remain unaltered. The field intensity at any point in the surrounding region due to the magnetic poles which may be induced by this current on any magnetic bodies in the vicinity will also be proportional to the strength of this current, *provided the permeabilities of these bodies are constant.* Under these conditions the directions of the resultant lines of force and the resultant lines of induction established by the electric current will remain unaltered when the strength of the current is changed, but their number crossing any surface will vary directly as the strength of the current. The lines of force and the lines of induction will also coincide in direction. Hence the number of lines of induction crossing any area in the magnetic field due to a current in a single circuit is directly proportional to the strength of the current in this circuit, provided all the surrounding bodies have a constant permeability.

In particular, the number of lines of induction threading the circuit itself is proportional to the strength of the current in the circuit. Therefore, calling i the strength of the current in the circuit at any instant, the number of lines of induction threading the circuit at this instant is proportional to i, and therefore the number of linkages λ between the circuit and the flux due to the current in it is proportional to i, that is

$$\lambda = Ai$$

where A is a constant depending upon the shape and size of the circuit and the magnetic nature of the bodies in the field, but independent of the strength of the current, *provided the permeability of every body in the field is constant.* Note that this constant A is equal to the number of linkages between this circuit and the

flux produced by unit current in it, for when $i=1$, $\lambda=A$. Moreover, from equation (15), this constant is the self inductance of the circuit, for $\frac{d\lambda}{di}=A$. The self inductance of a circuit may be looked upon as a constant which represents the extent and distribution of the magnetic field due a current in the circuit, but is independent of the value of the field intensity provided the permeabilities of all bodies in the field are constant.

Similarly, the number of lines of induction threading a given circuit 1 due to a current i_2 in another circuit 2 is proportional to the current in 2, and therefore the number of linkages λ_{12} of the circuit 1 by the flux due to the current in 2 is proportional to the current i_2, that is

$$\lambda_{12}=Bi_2$$

where B is a constant depending upon the size and shape of the two circuits, their relative positions and the nature of the bodies in the magnetic field, but independent of the strength of the current, *provided the permeability of every body in the magnetic field is constant.* Note that this constant B is equal to the number of linkages of the circuit 1 by the flux due to unit current in 2, for when $i_2=1$, $\lambda_{12}=B$. Moreover, from equation (16), this constant is the mutual inductance of circuit 2 with respect to circuit 1, for $\frac{d\lambda_{12}}{di_2}=B$. In the case of two circuits three constants are required to represent the extent and distribution of the magnetic field due to the currents in these circuits, the self inductance of each circuit and a single coefficient of mutual inductance (see Article 120).

118. Magnetic Energy of an Electric Current.— Since when the strength of the current in a circuit is increased a back electromotive force is induced in the circuit (due to the increase in the flux linking the circuit), a certain amount of energy is required to establish an electric current, just as energy is required to establish a water current, *i.e.*, to set a mass of water in motion. This energy comes from the source of the electromotive force which establishes the current, just as the energy for accelerating the velocity of a body comes from the source of the mechanical force which sets the body in motion. Since the magnetic field surrounding the circuit may be considered as the source of the back electromotive force in the circuit, the energy required to establish

the electric current may be said to be "stored" in the magnetic field, just as the kinetic energy of a moving body may be looked upon as stored in the body itself as a consequence of its inertia, which may be looked upon as the source of the opposition which the body offers to being set in motion. Again, when the strength of the current in a circuit decreases, an electromotive force is induced in the circuit in the *same* direction as the current, and therefore a certain amount of the energy stored in the magnetic field of the current is converted into some other form of energy by means of the current in the circuit (*e.g.*, heat energy), just as when the velocity of a moving body decreases a definite amount of its kinetic energy is converted into other forms of energy (*e.g.*, heat energy).

In general, no matter how a magnetic field may be formed, whether by means of electric currents or magnetic poles, a definite amount of energy is required to establish it and when the field is destroyed this energy appears in some other form. Every magnetic field therefore represents a certain amount of "stored," or potential, energy. This energy of a magnetic field is called *magnetic energy;* when the magnetic field is due to electric currents this energy is also called the *electrokinetic energy* of the currents, the latter name being due to the analogy between this energy and the kinetic energy of a moving body.

The magnetic or electrokinetic energy of the magnetic field set up by a current in a circuit may be expressed in terms of the current strength and the inductance of the circuit. Let i be the current in abamperes in the circuit at any instant and let L be the self inductance of the circuit, in abhenries; then the linkages λ between the circuit and the flux due to the current in it at this instant are $\lambda = Li$. The back electromotive force due to an increase di in the current in time dt is then $e = L\dfrac{di}{dt}$, *provided L is constant,* and therefore the energy transferred to the magnetic field by the current in time dt is

$$dW = eidt = Lidi$$

Therefore the total amount of energy stored in the magnetic field of the current when the strength of the current increases from 0 to any value i is

$$W = \int_0^i Lidi = \tfrac{1}{2}Li^2 \tag{17}$$

The energy stored in the field when the current is established is equal to the energy returned to the circuit when the current is interrupted, for

$$\int_i^o Lidi = -\int_o^i Lidi$$

provided L is constant. The energy returned to the circuit from the magnetic field gives rise to an electromotive force which tends to keep the current flowing in its original direction, just as when the motion of a body is opposed in any way, the kinetic energy of the body tends to keep it moving in its original direction.

The instantaneous value of this electromotive force at the instant a circuit is opened is generally sufficiently great to establish a momentary current through the air or whatever else separates the ends of the opened circuit, and consequently the current continues to flow for a fraction of a second across the space between the open ends of the circuit, producing the familiar *spark* or *arc* which occurs when a circuit is opened (unless certain special precautions are taken). The energy which produces this spark is the energy which is stored in the magnetic field around the circuit when the current is established, and is generally not sufficient to maintain for more than a fraction of a second a current through the high resistance of the insulator between the open ends of the circuit. However, whenever a current is established through an insulator its resistance falls to a comparatively low value, at which it remains as long as the current through it is maintained. Hence, when the gap formed by opening the circuit is short, the electromotive force originally in the circuit may be sufficient to maintain a comparatively large current through the gap. It is this heavy current established across the gap by the electromotive force originally in the circuit which constitutes the so-called arc, and which rapidly burns away the conductors at the gap unless a suitable switch or circuit-breaker is provided for opening the circuit. When the current in the circuit is large or the electromotive force in the circuit high, the circuit-breaker must open the circuit rapidly and make a long gap between the opened ends.

It should be noted that the above formula for the energy required to establish a magnetic field is deduced on the assumption that the self inductance is constant, which is true only when the permeability of every body in the magnetic field is constant. In the case of iron and other magnetic bodies the permeability

is not constant, and in addition to the transfer of energy to and from the magnetic field, a certain amount of energy is converted into heat energy (due to hysteresis, see Article 126), both when the current is established and also when the current is interrupted, or whenever there is any change whatever in the current strength.

119. Analogy between the Magnetic Energy of an Electric Current and the Kinetic Energy of a Moving Column of Water.—As already noted, the magnetic energy of an electric current is in many ways analogous to the kinetic energy of a moving column of any liquid, as for example, a current of water in a pipe. The kinetic energy of such a "water current" is the work required to set the water in motion; similarly, the magnetic energy of an electric current may be looked upon as the work required to "set the electricity in the circuit in motion." When the water comes to rest, its kinetic energy is converted into some other form, principally heat energy. Similarly, when an electric current is interrupted, thus causing the "electricity to come to rest," the magnetic energy of the current is converted into some other form, *e.g.*, heat energy in the conductors and in the spark which appears at the break in the circuit.

The kinetic energy of a moving column of water can also be expressed by a formula similar to the formula $W = \frac{1}{2} Li^2$ for the magnetic energy of an electric current. Consider a stream of water flowing through a pipe of uniform cross section of A sq. cm. and let the water completely fill the pipe; let l be the length in cm. of a given section of the pipe, and let V be the linear velocity of the water in cm. per second. Then, since the mass of 1 cu. cm. of water is 1 gram, the kinetic energy of the water in the given length of pipe is $\frac{1}{2}(lA)V^2$. Let J be the volume or "quantity" of water that flows across any section of the pipe in one second, then $J = AV$. Hence the kinetic energy of the water column may also be expressed as $\frac{1}{2}\frac{l}{A}J^2 = \frac{1}{2}KJ^2$, where $K = \frac{l}{A}$ is a constant depending upon the dimensions of the pipe. The analogy between this last expression and the formula $\frac{1}{2} Li^2$ for the magnetic energy of an electric current is at once evident, when an electric current is considered as the quantity of electricity per second flowing across any section of the wire. Note, however, that while the factor K in the expression for the kinetic energy of the water current

depends only upon the length and cross section of the given length of pipe, the factor L in the expression for the magnetic or electrokinetic energy of an electric current depends not only upon the length and cross section but also upon the shape of the wire forming this circuit, upon the shape of the circuit, and upon the nature of the surrounding bodies.

120. Magnetic Energy of Two or More Electric Currents.—As already noted (Article 116), when the strength of the current in an electric circuit changes the corresponding change in its magnetic field induces an electromotive force not only in this circuit but also in every other circuit in the vicinity. Consequently if a current already exists in any neighboring circuit, or if the induced electromotive force establishes a current in such a circuit, there will be a transfer of energy from one circuit to the other. Whether work is done on the current or is done by the current in a given circuit depends upon the relative direction of the current and the electromotive force induced in this circuit. In any problem dealing with the mutual effects of two or more circuits it is therefore necessary to adopt some convention in regard to the algebraic signs of the various currents; this is conveniently done by choosing a given sense of the lines of induction as positive (*e.g.*, left to right) and to consider the current in any circuit as positive if the lines of induction which it sets up thread this circuit in this same sense, negative if these lines thread the circuit in the opposite sense. For example, when two coils of wire are placed side by side, the currents in the two coils are to be considered in the same sense if the lines of induction set up by the current in one coil thread the other in the same direction as the lines of induction set up by the current in the latter.

Consider first the simple case of two circuits in the vicinity of each other, and let the circuits be fixed in size, shape and relative position, and let the permeability of every body in the field be constant. Let L_1 and L_2 be the self inductances of the two circuits and M_{12} the mutual inductance of circuit 2 with respect to circuit 1 and M_{21} the mutual inductance of 1 with respect to 2; let i_1 and i_2 be the currents in the two circuits at any instant. Then, from Article 117, the total linkages of the two circuits by the flux threading them are respectively

$$\begin{aligned} \lambda_1 &= L_1 i_1 + M_{12} i_2 \\ \lambda_2 &= L_2 i_2 + M_{21} i_1 \end{aligned} \qquad (18)$$

where the λ's and i's may be positive or negative. Let the two currents increase by the amounts di_1 and di_2 in time dt; then the back electromotive forces induced in the two circuits are respectively

$$
\begin{aligned}
e_1 &= \frac{d\,\lambda_1}{dt} = L_1 \frac{di_1}{dt} + M_{12} \frac{di_2}{dt} \\
e_2 &= \frac{d\,\lambda_2}{dt} = L_2 \frac{di_2}{dt} + M_{21} \frac{di_1}{dt}
\end{aligned}
\qquad (19)
$$

and the amounts of energy stored in the magnetic field by the respective currents are

$$
\begin{aligned}
dW_1 &= e_1 i_1 dt = L_1 i_1 di_1 + M_{12} i_1 di_2 \\
dW_2 &= e_2 i_2 dt = L_2 i_2 di_2 + M_{21} i_2 di_1
\end{aligned}
$$

The total amount of energy stored in the magnetic field by two currents when they increase from zero to their final values I_1 and I_2 may be calculated as follows. First, let circuit 2 be open, so that no current can flow in it; under these conditions $i_2 = 0$ and $di_2 = 0$, and therefore the work done by the current in circuit 1 when the current in this circuit increases from zero to I_1 is

$$\int_0^{I_1} L_1 i_1 di_1 = \tfrac{1}{2} L I^2_1$$

Now keep the current in circuit 1 constant and let the current in 2 increase from 0 to I_2; under these conditions $i_1 = I_1$ and $di_1 = 0$, and therefore the work done by the current in circuit 1 is

$$\int_0^{I_2} M_{12} I_1 di_2 = M_{12} I_1 I_2$$

and the work done by the current in circuit 2 is

$$\int_0^{I_2} L_2 i_2 di_2 = \tfrac{1}{2} L_2 I^2_2$$

Hence the total work done by the two currents in establishing the magnetic field corresponding to the final values of the currents I_1 and I_2 is

$$W = \tfrac{1}{2} L_1 I^2_1 + M_{12} I_1 I_2 + \tfrac{1}{2} L_2 I^2_2 \qquad (20a)$$

Note that this formula does not contain the coefficient M_{21}. The explanation of this is the fact, already noted several times, that the mutual inductance of one circuit with respect to another is the same as the mutual inductance of the second circuit with respect to the first circuit. To prove this, let the current in 2 now be kept constant and let the current in 1 be decreased to zero;

under these conditions $i_2 = I_2$ and $di_2 = 0$; and therefore the work done on the current in 1 by the magnetic field is

$$-\int_{I_1}^{o} L_1 i_1 di_1 = \tfrac{1}{2} L_1 I^2_1$$

and the work done on the current in 2 by the magnetic field is

$$-\int_{I_1}^{o} M_{21} I_2 di_1 = M_{21} I_1 I_2$$

Now open circuit 1 and let the current in 2 fall to zero; under these conditions $i_1 = 0$ and $di_1 = 0$, and therefore the work done on the current in 2 by the magnetic field when this current falls to zero is

$$-\int_{I_2}^{o} L_2 i_2 di_2 = \tfrac{1}{2} L_2 I^2_2$$

Hence the total work done by the magnetic field when it disappears is

$$W = \tfrac{1}{2} L_1 I^2_1 + M_{21} I_1 I_2 + \tfrac{1}{2} L_2 I^2_2 \qquad (20b)$$

From the Principle of the Conservation of Energy, the energy *stored* in the field when it is established must be equal to the work done by the field when it disappears, hence the expression for the energy stored in the field must be equal to the expression for the energy given back by the field; therefore

$$M_{21} = M_{12}$$

It also follows from the Principle of the Conservation of Energy that the total amount of energy stored in the magnetic field by the two currents is independent of the manner in which these two currents are established, and therefore the expression

$$W = \tfrac{1}{2} L_1 i_1^2 + M i_1 i_2 + \tfrac{1}{2} L_2 i^2_2 \qquad (21)$$

is a perfectly general one for the energy of the magnetic field due to the currents i_1 and i_2 in two circuits which have constant self inductances L_1 and L_2 and a constant mutual inductance M.

Equation (21) may also be written

$$W = \tfrac{1}{2} (L_1 i_1 + M_{12} i_2)\, i_1 + \tfrac{1}{2} (L_2 i_2 + M_{21} i_1)\, i_2$$

which, in turn, from equation (18), may be written

$$W = \tfrac{1}{2} \lambda_1 i_1 + \tfrac{1}{2} \lambda_2 i_2$$

By exactly similar reasoning it can be shown that the energy of the magnetic field due to the electric currents in any number of circuits is

$$W = \tfrac{1}{2} \Sigma \lambda i \qquad (22)$$

where i is the current in any circuit and λ is the number of linkages between this circuit and the total flux which threads it, and the summation includes all the circuits in the field.

It should be noted that all the formulas in this article are based upon the assumption that every body in the magnetic field has a constant permeability.

121. Calculation of Inductance. — Skin Effect. — When the permeability of every body in the magnetic field is constant, it is in general possible to obtain an expression for the inductance of a circuit in terms of the dimensions of the circuit. To obtain an exact expression for this quantity, however, it is necessary to consider the conductors forming the circuit as divided into current filaments (see Article 101) and to determine the back electromotive force induced in each of these filaments when the current changes. The inductance of a circuit therefore depends upon the distribution of the current in the conductors, and, as already noted (Article 101), the distribution of the current in a conductor depends upon the rapidity with which the current varies with time. The explanation of this will now be apparent; when the conductor has a large cross section or is a magnetic substance (*e.g.*, a steel rail) the lines of induction within the substance of the conductor are an appreciable proportion of the total number of lines of induction which link the conductor; moreover, since a line of induction which lies wholly within the substance of the conductor links only that portion of the current which threads the loop formed by this line, it follows that the number of lines of induction which link the inside filaments is greater than the number which link the outside or surface filaments, and consequently the back electromotive force induced in the inside filaments is greater than that induced in the outside filaments. The result of this is that a greater proportion of the current flows through the outside filaments than through the inside filaments, *i.e.*, the current density is greatest near the surface of the conductor. In certain simple cases the exact distribution of the current for a given impressed electromotive force can be determined by expressing the condition that the impressed electromotive force acting on each filament must be equal to the sum of the resistance drop and the back electromotive force induced in that filament. This phenomenon of a non-uniform current distribution caused by a rapid variation of the current with respect to time is called the "skin effect." As noted in Chapter VII, this skin effect causes an increase in the apparent resistance of the circuit; the self inductance of a circuit is diminished by the skin effect.

In the case of wires of the size ordinarily employed in practice the skin effect may be neglected and the current density assumed to be constant over the cross section of the wire provided the current varies comparatively slowly with time. (In the case of an alternating current the frequency may be as high as 60 cycles per second provided the conductor is non-magnetic; the skin effect in a steel rail, however, is quite appreciable even for low frequencies.) The assumption of no skin effect is equivalent to assuming that the back electromotive force induced in each filament of the conductor is the same, and therefore that the number of linkages of the conductor as a whole may be taken as the *average* of the linkages of all the filaments taken separately. Approximate formulas for self inductance may also be obtained by neglecting altogether the lines of induction inside the substance of the conductor.

122. Self Inductance of Two Long Parallel Wires. — As an example of the approximate method of calculating inductance, consider a circuit formed by two non-magnetic parallel wires, when the wires are so long that the magnetic field due to the current in the conductors connecting their ends may be neglected.

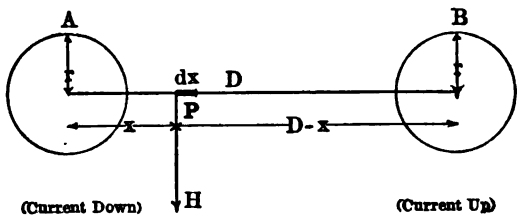


Fig. 73.

Let the wires be of the same size and have a radius of r centimeters, and let them be spaced D centimeters between centers. Let the current in each wire be i abamperes; away from the reader in A (Fig. 73) and toward the reader in B. Then the field intensity at any point P at a distance x from the center of the wire A on the line between A and B is (see equation 4a of Chapter III).

$$H=\frac{2i}{x}+\frac{2i}{D-x}$$

Hence, the number of lines of induction crossing an area at P of width dx and unit length parallel to the two wires is, since the

field intensity is perpendicular to this plane and the permeability of the air surrounding the wires is unity,

$$d\,\phi = H\,dx = 2I\left[\frac{dx}{x}+\frac{dx}{D-x}\right]$$

Hence, the total number of lines of induction per centimeter length of the wires threading the space between the two is

$$\phi = \int_{x=r}^{x=D-r} d\,\Phi = 2I\Big[\,ln\,x - ln\,(D-x)\,\Big]_{x=r}^{x=D-r} = 4I\,ln\,\frac{D-r}{r}$$

(The abbreviation "*ln*" is used for the natural logarithm and the abbreviation "log" for the common logarithm.) Whence the inductance of one centimeter length of *both* wires is

$$L = \frac{\phi}{I} = 4\,ln\,\frac{D-r}{r} \text{ abhenries}$$

Or, since each wire is linked by *half* the resultant lines of induction linking the space between the two, the inductance of *each* wire per unit length is

$$L = 2\,ln\,\frac{D-r}{r} \text{ abhenries}$$

These formulas, however, are only approximate, since the lines of induction *inside* the wires have been neglected. It can be shown (see Alex. Russell, *Alternating Currents*, Vol. 1, p. 55) that the exact formula for the inductance of *each* of two non-magnetic parallel wires for slowly varying currents is

$$L = 0.5 + 2\,ln\,\frac{D}{r} \qquad \text{abhenries per cm.} \qquad (23a)$$

or

$$L = 0.01524 + 0.1403\,log\,\frac{D}{r} \quad \text{millihenries per 1000 ft.} \qquad (23b)$$

or

$$L = 0.08047 + 0.7411\,log\,\frac{D}{r} \quad \text{millihenries per mile} \qquad (23c)$$

It should be noted that since D and r occur in these formulas only as a ratio, it is immaterial in what units D and r be expressed, provided they are both expressed in the *same* unit.

The minimum value of the inductance is when the wires touch. In this case $D = 2r$, and therefore $L = 0.0575$ millihenries per

1000 feet and is independent of the size of the wire. An absolutely non-inductive circuit is impossible, although this condition may be closely approximated by placing the wires forming the circuit close together, *e.g.*, by twisting them together.

123. Self Inductance of a Concentrated Winding. — Let L_1 be the self inductance of a single loop of wire of any shape; then the flux linking this loop due to unit current in it is $\phi = L_1$. Consider a coil made of N turns of wire, each turn of exactly the same dimensions as this single loop and let these N turns be so close together that they may all be considered as coinciding exactly with one another, *i.e.*, let the N turns be considered as concentrated in a geometrical line. Then unit current in each of these turns will set up a flux ϕ which will link each of the N turns; therefore the total flux linking each turn will be $N\phi = NL_1$, and since there are N turns the number of linkages between the total flux and the entire coil will be N^2L_1, which, from Article **117**, is equal to the self inductance L_n of the entire coil, *i.e.*,

$$L_n = N^2 L_1 \qquad (24)$$

Hence the self inductance of a *concentrated* winding is proportional to the *square* of the number of turns in it.

124. Self Inductance of a Long Solenoid. — In contradistinction to a concentrated winding, consider the case of a long air-core solenoid, which is one form of a distributed winding. Let N be the number of turns in the solenoid, l its length in centimeters and A its mean cross section in square centimeters. Then, from equation (4), the magnetic field intensity at any point inside the solenoid at a considerable distance from its ends due to a current of one abampere in the coil is $H = \frac{4 \pi N}{l}$ and is parallel to the axis of the solenoid. The total number of lines of induction linking each of the central turns (neglecting the lines within the substance of the wire) is therefore

$$\phi = HA = \frac{4 \pi NA}{l}$$

The flux linking the end turns is less than this, since some of the lines of induction go through the lateral walls of the solenoid; as an approximation, however, in the case of a long solenoid all the lines of induction may be assumed to link these end turns also. On this assumption the total number of linkages between the coil

and the flux per unit current in the coil, *i.e.*, the self inductance of the coil, is then

$$L = N\phi = \frac{4\pi N^2 A}{l} \tag{25}$$

where all the quantities are in *c. g. s.* units.

The self inductance of a solenoid, therefore, does not vary with the square of the number of turns when the variation is made by changing its length, for in this case the number of turns varies as the length, and therefore the inductance varies directly as the number of turns, or directly as the length. Hence when a solenoid is used as a variable inductance, by connecting in the circuit a greater or less number of its turns by means of a sliding contact, for example, the inductance varies directly as the distance between the fixed terminal and the slider, provided this distance is large compared with the diameter of the solenoid. When the slider is close to the fixed terminal, the turns between the two form an approximately concentrated winding, and therefore the inductance when only a small part of the winding is used varies approximately as the square of the distance between the two terminals.

It should be noted that when a coil of any kind is used as a part of an electric circuit, there is always a mutual inductance between the rest of the circuit and the coil. This mutual inductance, however, may be made practically negligible by making the leads to the coil sufficiently long and twisting them together.

125. Total Energy of a Magnetic Field in Terms of the Field Intensity and Flux Density. — The total energy of the magnetic field due to any number of electric currents may also be expressed in terms of the field intensity and the flux density. Consider the case when every body in the magnetic field has a constant permeability; the flux density and field intensity at every point in the field will then be proportional to each other and in the same direction. Under these conditions the energy corresponding to the final values of the currents must be independent of the manner in which these currents are established; we may then for convenience assume that the currents all increase proportionally from zero to their final values. Under these conditions, if we imagine all space filled with tubes drawn in such a manner that the walls of each tube are tangent at each point to the direction of the flux density at that point due to the *final* values of the currents, then

the walls of these tubes will also be tangent to the flux density at any instant while the currents are rising from zero to their final values. These tubes will also be closed on themselves and from equation (8) the current linked by each tube at any instant will be $i=\frac{1}{4\pi}\int_{|L|} Hdl$, where dl is any elementary length measured along the axis of the tube and H the intensity of the field at dl. When all the currents change by a certain small amount, producing a change of flux density dB at every point in the field, the change in the flux through each tube will be $d\phi = dB.ds$ where ds represents any section of the tube at right angles to its axis, and dB the change in the flux density at this point. dB and ds will of course vary from point to point in the field but, from the manner in which these tubes are drawn, the *product $dB.ds$ will be constant for every section of any one tube.* The work done in changing the flux through each tube by an amount $d\phi$ is

$$i\,d\phi=\frac{1}{4\pi}\,dB.ds\int_{|L|} Hdl$$

and the total work done corresponding to all the tubes is the volume integral

$$\frac{1}{4\pi}\int_{|S|} dB.ds\int_{|L|} Hdl$$

throughout all space. But, *since $dB.ds$ is constant along any tube,* this may be written

$$\frac{1}{4\pi}\int_{|S|}\int_{|L|} HdB\; ds.dl=\frac{1}{4\pi}\int_{|V|} HdB\; dv$$

since $ds.dl$ represents an elementary volume dv at each point. Hence the work done *per unit volume of space,* when the currents in the various circuits change proportionally by an infinitesimal amount, is

$$dw=\frac{1}{4\pi}\,HdB$$

Hence the *total work done per unit volume of space* by any number of currents in establishing their resultant magnetic field is

$$w=\frac{1}{4\pi}\int_0^B HdB=\frac{\mu H^2}{8\pi}=\frac{B^2}{8\pi\mu} \qquad (26)$$

provided the permeability of every body in the field is constant.

When the permeability of the bodies in the field is not constant, but varies with the field intensity, the reasoning employed in deducing the expression $\frac{1}{4\pi}\int_0^B HdB$ for the work done by the currents per unit volume of the field, breaks down; for in this case the direction of the tubes which we considered as filling all space may change as the currents change in value. However, the expression $\frac{1}{4\pi}\int_0^B HdB$ for the energy per unit volume required to establish the field (only a portion of this, however, is *stored* in the field, see Article 126*a*) is applicable to any case where the direction of the lines of induction remains unaltered and coincides with, or is in the opposite direction to, the lines of force. This condition is realized in the special case of a closed anchor ring which is magnetised by a coil uniformly wound around it (as described in Article 110) and is also approximately realized in the magnetic circuits of most electrical apparatus.

126. Heat Energy Due to Hysteresis. — When the permeability of every body in the magnetic field is constant, the energy transferred to the magnetic field when the flux density increases from zero to any value B is exactly equal to the energy transferred from the magnetic field to the electric currents when the flux density decreases from B to zero; for, when B is proportional to H, $\frac{1}{4\pi}\int_0^B HdB = -\frac{1}{4\pi}\int_B^0 HdB.$ However, in the case of ordinary magnetic bodies such as iron or steel, we have already seen (Article 57) that the relation between B and H when the flux density increases is different from the relation between these two quantities when the flux density decreases. Hence in this case $\frac{1}{4\pi}\int_0^B HdB$ *is not equal to* $-\frac{1}{4\pi}\int_B^0 HdB.$ The first expression, however, represents the work done *by* the currents in establishing themselves, while the second expression represents the work done *on* the currents when the magnetic field disappears; the difference between these two expressions must then represent the production of some other form of energy in the magnetic substance, and experiment shows that this other form of energy is heat energy. That is, whenever the flux density in a piece of any

ordinary magnetic substance is changed, heat energy is always developed. Consequently, when the flux density in a piece of iron or steel is changed from any value B to any other value and then brought back again to the original value B, an amount of heat energy per unit volume of the substance equal to

$$w=\frac{1}{4\pi}\int_{|B|} HdB \qquad (27)$$

for this complete cycle of values, is produced in the substance. Hence, when this cycle of values is plotted (with H and B to the *same* scale) in a curve (*i.e.*, the hysteresis curve), the area of this curve divided by 4π gives the amount of heat energy *per unit volume* "due to hysteresis" produced in the substance. When H and B are expressed in *c. g. s.* electromagnetic units, this heat energy is in ergs per cubic centimeter.

126a. Tractive Force of an Electromagnet. — From the formula, equation (26), for the energy per unit volume of a magnetic field, can be deduced directly the force of attraction between two electromagnets or between an electromagnet and its keeper. Consider the special case where the surface of separation of the two parts of the magnetic circuit is perpendicular to the lines of induction; let S be the area of this surface in square centimeters and let dx be the infinitesimal amount by which the two parts of the magnetic circuit are separated. When dx is infinitely small, the change produced in the reluctance of the circuit is negligible, and therefore the flux density in the air gap formed is the same as originally existed in the iron; let this flux density be B and let it be assumed constant over the surface of separation. The volume of the magnetic field is changed by an amount Sdx, and consequently the energy of the magnetic field, from equation (26), is changed by the amount

$$dW=\frac{B^2}{8\pi}Sdx$$

But the work done in separating the two parts of the circuit is also equal to the product of the displacement dx, and the mechanical force F in dynes required to produce the separation, *i.e.*,

$$dW=Fdx$$

Equating these two expressions, we have

$$F=\frac{B^2S}{8\pi} \qquad (28)$$

When the flux density is not constant over the surface of separation, the total force required is the surface integral

$$F = \frac{1}{8\pi}\int_S B^2 ds \qquad (28a)$$

SUMMARY OF IMPORTANT DEFINITIONS AND PRINCIPLES

1. The number of **linkages** between a line of induction and a coil of wire is equal to the number of turns of this coil linked by the line of induction. The total number of linkages between a coil and the flux threading it is the sum of the linkages of all the lines of induction. For a **concentrated winding** of N turns and each turn linked by ϕ lines, the number of linkages is

$$\lambda = N\phi$$

Linkages are expressed in mawells.

2. Whenever the flux threading a circuit changes an **electromotive force is induced** in this circuit equal to the time rate of change of the linkages between the flux and the circuit, *i.e.*,

$$e = -\frac{d\lambda}{dt} \quad \text{abvolts} = -10^{-8}\frac{d\lambda}{dt} \quad \text{volts}$$

where λ is in maxwells and t in seconds. The minus sign in the above equation indicates that the direction of the electromotive force resulting from an increase in the flux in the same direction as that of the flux due to the current in the circuit produces an electromotive force in the opposite direction to that of the current.

3. The **work done on an electric current** when the number of linkages between the circuit and the flux linking it increases by an amount $d\lambda$ is

$$dW = -i d\lambda \quad \text{ergs}$$

where i is in abamperes and λ in maxwells.

4. When the change in the linkages between an electric circuit and the flux linking it is caused by the **motion** of a part of this circuit (*e.g.* a wire) **through a magnetic field,** the induced electromotive force is equal to the rate at which this part of the circuit cuts the lines of induction. The direction of the electromotive force is the direction in which the middle finger of the **right hand** points when laid along the wire and the thumb and forefinger of this hand are held perpendicular to each other and to the middle finger, with the thumb pointing in the direction of the

motion of the wire and the forefinger in the direction of the flux density. The value of the electromotive force induced in a wire l centimeters long when it moves perpendicularly across a magnetic field of flux density of B gausses with a velocity of v centimeters per second perpendicular to itself is

$$e = Blv \qquad \text{abvolts}$$

5. The **intensity of the magnetic field** due to a current of i abamperes in a **long solenoid** of N' turns per centimeter length is

$$H = 4\pi N'i \text{ gilberts per cm}$$

6. The **quantity of electricity discharged** through a ~~circuit of R~~ abohms resistance when the number of **linkages** between the circuit and the flux threading it **changes** from λ_1 to λ_2 maxwells is

$$Q = \frac{\lambda_1 - \lambda_2}{R} \qquad \text{abcoulombs}$$

7. The average value of the induced electromotive force between the brushes of a continuous current dynamo is

$$E = \frac{np\phi N}{p' \times 10^8} \qquad \text{volts}$$

where n is the number of revolutions per second, p the number of field poles, p' the number of parallel paths through the armature, ϕ the useful flux per pole in maxwells, and N is the number of conductors which cut the flux.

8. The work required to carry a unit north pole around a closed path is called the **magnetomotive force** acting around this path. The magnetomotive force acting around a closed path which links N turns of a circuit in which the current is i abamperes is

$$4\pi N i \qquad \text{gilberts}$$

The product of the current in amperes by the number of turns is called the **ampere-turns** of the coil.

$$1 \text{ gilbert} = 0.79578 \text{ ampere-turn}$$

9. The **reluctance** of a given portion of a magnetic circuit is defined as the ratio of the drop of magnetic potential along the lines of induction to the total number of lines of induction through the given portion of the circuit. The *c. g. s.* electromagnetic unit of reluctance is the oersted. The reluctance of a portion of a circuit l centimeters long, A square centimeters in cross section and of a permeability μ is

$$R = \frac{l}{\mu A} \qquad \text{oersteds}$$

provided the flux density is uniform and perpendicular to the cross section.

10. The **ampere-turns required** to establish a given flux are found by equating the drop of magnetic potential through the magnetic circuit to the magnetomotive force, *i.e.*,

$$\int_{|L|} H\, dl = NI$$

where H is in ampere-turns per inch, l is in inches, I in amperes, and the integral is taken around the closed path formed by the magnetic circuit.

11. The value of the ratio of the electromotive force induced in an electric circuit, due solely to a change of the current in this circuit, to the time rate of this change, is called the **self inductance** L of this circuit; hence the self-induced electromotive force is

$$e = L\frac{di}{dt}$$

The *c. g. s.* unit of self inductance is the abhenry; the practical unit is the henry.

$$1 \text{ henry} = 10^9 \text{ abhenries}$$

When the permeability of every body in the magnetic field is constant and the circuit remains unaltered in shape, the self inductance is a constant of the circuit (for a given distribution of current) equal to the number of linkages between this circuit and the flux produced by unit current in it.

12. The value of the ratio of the electromotive force induced in an electric circuit 1, due to a change of the current in any other circuit 2, to the time rate of change of this current, is called the **mutual inductance** M_{12} of circuit 2 with respect to circuit 1; hence the electromotive force induced in 1 due to a change of the current i_2 in 2 is

$$e_{12} = M_{12}\frac{di_2}{dt}$$

The units of mutual inductance are the same as the units of self inductance. When the permeability of every body in the magnetic field is constant and the circuits are fixed with respect to each other and remain unaltered in size and shape, the mutual inductance of one circuit with respect to the other is constant (for a given distribution of the currents) and equal to the number of linkages between one circuit and the flux produced by unit current in the other, and the mutual inductance of the one circuit with

respect to the other is the same as the mutual inductance of the second circuit with respect to the first.

13. The **energy of the magnetic field due to a current** i in a circuit of constant self inductance L is

$$W = \tfrac{1}{2} Li^2$$

When L is in henries and i in amperes this formula gives the energy in joules; when L is in abhenries and i in abamperes this formula gives the energy in ergs.

14. The **energy of the magnetic field due to the currents** i_1 **and** i_2 **in two circuits** which have constant self inductances L_1 and L_2 and a constant mutual inductance M is

$$W = \tfrac{1}{2} Li^2{}_1 + Mi_1i_2 + \tfrac{1}{2} Li^2{}_2$$

When the inductances are in henries and the currents in amperes this formula gives the energy in joules; when these quantities are in abhenries and abamperes this formula gives the energy in ergs.

15. When the current in a conductor of large cross section varies rapidly with time, the back electromotive force due to the variation of the flux linking the conductor is considerably greater in the inside filaments of the conductor than in the outside filaments; consequently the current density is greatest near the surface of the conductor. This phenomenon is known as the **skin effect.**

16. The **self inductance** of each of two long **parallel** non-magnetic **wires,** assuming a uniform current density, is

$$L = 0.01524 + 0.1403 \log \frac{D}{r} \qquad \text{millihenries per mile}$$

where D is the distance between centers and r the radius of each wire, both in the same units.

17. The **self inductance** of a **concentrated winding** varies directly as the square of the number of turns.

18. The **self inductance** of a **long solenoid** is approximately

$$L = \frac{4\pi N^2 A}{l} \qquad \text{abhenries}$$

where N is the total number of turns in the solenoid, A its cross section in square centimeters and l its length in centimeters.

19. The energy per cubic centimeter of any magnetic field when the permeability of every body in the field is constant is

$$w = \frac{\mu H^2}{8\pi} = \frac{B^2}{8\pi\mu} \qquad \text{ergs}$$

where H is the field intensity in gilberts per centimeter, B the flux density in gausses and μ the permeability in *c. g. s.* electromagnetic units.

20. The **heat energy dissipated** per cycle per centimeter-cube in a **magnetic substance** when an alternating magnetic field is established in it is equal to the area of the corresponding hysteresis loop divided by 4π.

21. The force required to separate two parts of a magnetic circuit is

$$F=\frac{B^2A}{8\pi} \qquad \text{dynes}$$

where A is the area in sq. cm. of the surface of separation and B the flux density in gausses normal to this surface, provided B is constant.

PROBLEMS

1. A coil which has a concentrated winding of 100 turns is revolved in a uniform magnetic field about an axis perpendicular to the direction of the field. The intensity of the field is 100 gilberts per cm., the area of each turn of the coil is 10 sq. cm.; the coil is rotated with a uniform velocity of 25 revolutions per second. What is the instantaneous value of the *e. m. f.* induced in the coil, (1) when its plane is perpendicular to the field, (2) when its plane is parallel to the field; (3) what is the average value of the *e. m. f.* induced in the coil while it is rotating 180° from the first position?

Ans.: (1) 0; (2) 1.013 volts; (3) 0.645 volts.

2. How many foot-pounds of work must be done to pull a coil having a concentrated winding of 200 turns out of a magnetic field, if the flux threading the coil is 100,000 maxwells and the current in it is maintained constant at 100 amperes?

Ans.: 14.76 foot-pounds.

3. A straight wire 10 inches long is moved parallel to itself across a magnetic field in a plane making an angle of 45° with the direction of the field. What is the value of the *e. m. f.* in volts induced in the wire if it is moved with a velocity of 5 feet per second and the field intensity is 200 gilberts per cm.?

Ans.: 0.00548 volts.

4. A coil of 20 turns is wound over the middle of a long solenoid with an air core. The solenoid has a cross-sectional area of 10 sq. cm. and has 50 turns per linear inch. If the 20-turn secondary

is connected in series with a resistance such that the total resistance of the secondary circuit is 1000 ohms, what quantity of electricity in coulombs is discharged through the secondary circuit when a current of 5 amperes is reversed in the primary?

Ans.: 4.95×10^{-7} coulombs.

5. A coil of 1000 turns is wound uniformly on a round iron bar 20 centimeters in length and 2 sq. cm. in cross section. If a constant current of 5 amperes is established in the coil, calculate the work in joules required to withdraw the bar from the coil, assuming the permeability of the bar to be constant and equal to 400 *c. g. s.* units and neglecting the demagnetising effect of the ends of the bar.

Ans.: 0.01252 joules.

6. An air-core solenoid of 1000 turns is 20 cm. long and has a mean diameter of 5 cm. Calculate the field intensity on the axis of the solenoid 5 cm. internally from one end, due to a current of 2 amperes in the solenoid; make this calculation by integrating the effect due to each turn separately. What is the error involved in using formula (4) for the field intensity at this point?

Ans.: 122.8 gilberts per cm. Formula (4) gives a value 20.8% too great.

7. A cast-iron ring 20 inches in mean diameter with a circular cross section 2 inches in diameter has an air gap 0.1 inch in length. (1) How many ampere-turns are required to produce an average flux density of 5000 gausses in this air gap, assuming no leakage? Use Fig. 38 to get the relation between flux density and field intensity in the iron. (2) What is the percentage ratio of the drop of magnetic potential in the gap to that in the iron?

Ans.: (1) 4090 ampere-turns; (2) 32.8%.

8. What is the total reluctance of the iron ring and air gap described in Problem 7?

Ans.: 0.159 oersteds.

9. An iron ring 20 cm. in mean diameter and 4 sq. cm. in cross section is wound uniformly with 2000 turns of wire. Determine the self inductance of this winding in henries, assuming a constant permeability of 500 *c. g. s.* units.

Ans.: 1.6 henries.

10. Two coils A and B are wound uniformly upon the same magnetic circuit, one over the other. The self inductance of A is 3 henries and the self inductance of B is 5 henries. Assuming the permeability of the magnetic circuit to be constant, determine

the total inductance of A and B when connected in series, (1) so that their magnetic fields are in the same direction, and (2) so that their magnetic fields are in opposite directions. (3) What is the mutual inductance of A and B?

Ans.: (1) 15.74 henries; (2) 0.26 henries; (3) 3.87 henries.

11. A coil having a concentrated winding of 1000 turns has a self inductance of 0.05 henries. What is the energy in joules of the magnetic field due to the current in the coil when this current produces a flux of 2000 maxwells?

Ans.: 0.004 joule.

12. When a current of 4 amperes is established in the field winding of a 4-pole shunt generator, the flux of induction through each field coil is 3×10^6 maxwells. The four field coils are connected in series electrically and each has 800 turns. Assuming constant permeability and no magnetic leakage, determine (1) the self inductance of the entire field winding, and (2) the total energy of the magnetic field of the generator.

Ans.: (1) 24 henries; (2) 384 joules.

13. A cast-iron ring 10 inches in mean diameter and 4 sq. in. in cross section is divided into two halves. Upon each half of the ring are wound uniformly 100 turns of wire. If the two halves of the ring are placed together to form a complete ring and the two coils are connected in series so that their magnetic fields add, determine (1) the energy of the magnetic field per cubic centimeter when the current established in the winding is 2 amperes, assuming a constant permeability of 300 *c. g. s.* units, and (2) the initial force in pounds required to pull the two halves of the ring apart.

Ans.: (1) 473 ergs; (2) 16.5 pounds.

V

ELECTROSTATICS

127. Electric Charges. — We have already seen (Article 64) that it is possible to have a difference of electric potential between two conductors without having an electric current in these conductors. For example, the difference of electric potential between the two poles of a battery when there is no conductor connecting its poles, that is, when the battery is "open-circuited," is equal to the electromotive force of the battery. It is found by experiment that when a conductor A of any kind is connected to one of the poles of a battery but not to the other, a momentary electric current is established in this conductor, but after a small fraction of a second this current ceases. By Ohm's Law, every point of this conductor must come to the same potential as the terminal of the battery to which it is connected (neglecting any slight contact electromotive force between the conductor and the terminal). Similarly, in any other conductor B connected to the other terminal of the battery but completely insulated from the first terminal, a momentary electric current will be established, but when this current ceases every point of the conductor B will be at the same potential as that of the terminal to which it is connected. Hence between the two conductors A and B a difference of potential is established equal to that of the electromotive force of the battery. It is also found by experiment, that the wires connecting the two conductors A and B to the battery may be removed without changing this difference of electric potential between them, provided the wires are small and the position of the conductors A and B relative to each other and all other bodies in their vicinity remains fixed and the conductors remain perfectly insulated from each other and all other conductors. This can be tested by again connecting the conductors A and B to the same terminals of the battery; it will be found that no current, not even a momentary current, is established in the wires, thus showing that each conductor remains at the potential of the terminal to which it was originally con-

nected. A difference of electric potential may then exist between two conductors which are entirely insulated from each other and all other conductors.

It is found by experiment that whenever a difference of electric potential is established between any two conductors these conductors exert a force upon each other and in general also upon all other conductors and dielectrics in their vicinity, even when there are no currents in the conductors. These forces are due solely to the difference of electric potential established between the various bodies, and as long as these potential differences remain constant in value the forces are found to remain constant. Just as the forces produced by magnets on one another and upon magnetic substances may be expressed in terms of a something associated with the surfaces of the magnets and magnetic bodies, so may these forces due to differences of electric potential be expressed in terms of a something associated with the surfaces of the conductors and the dielectrics in their vicinity. The something that produces these forces which are due to differences of electric potential is called an *electric charge.* We wish now to see what are the properties of these electric charges and what is their relation to an electric current.

In the first place, it is found that forces of exactly the same nature are produced between any two dielectrics, such as glass and silk, when these two bodies are rubbed together and then separated; also when an insulated conductor is rubbed with any dielectric, such as a piece of silk, and the conductor and the dielectric are separated, it is found that the two bodies attract each other. Bodies which are charged in this way are said to be "charged by friction." In fact, this method of producing the phenomena which are ascribed to electric charges was known to the ancients; whereas batteries and generators are comparatively recent inventions.

128. Positive and Negative Charges. — Attraction and Repulsion of Charged Bodies. — It is found by experiment that bodies which are charged in exactly the same manner, as for example, two insulated conductors placed momentarily in contact with the same terminal of a generator, always *repel* each other, while they may repel or attract a body which has been charged in some other manner. For example, an insulated conductor A placed momentarily in contact with the positive

terminal of a generator is found to repel a body B which has been placed momentarily in contact with this terminal, but will attract the body B if the latter is placed in contact with the negative terminal of the generator. (These forces are perceptible only in case the generator develops an *e. m. f.* of several thousand volts or more.) Similarly, two pieces of glass which have been rubbed with silk are found to repel each other, while the silk and glass are found to attract each other. We are therefore led to the conception of two kinds of electric charges, which are called respectively positive and negative charges. A body which *repels* a piece of glass which has been rubbed with silk is said to be *positively* charged, or to "have" a *positive* charge, while a body which *attracts* a piece of glass which has been rubbed with silk is said to be *negatively* charged or to "have" a *negative* charge. A conductor which is placed in contact with the positive pole of a battery or other source of electromotive force is positively charged, while a body placed in contact with the negative pole of a battery or other source of electromotive force is negatively charged.

129. Charging by Contact and by Induction. — It is found by experiment that when an originally uncharged conductor U (*e.g.*, a solid piece of metal, not necessarily a wire) is placed in contact with a charged conductor C, the uncharged conductor likewise receives a charge which is of the same sign as that of the charged conductor; that is, when the charged conductor C is positively charged, the originally uncharged conductor U placed in contact with it likewise becomes positively charged; while if the originally uncharged conductor U is placed in contact with a negatively charged conductor C', the uncharged conductor U becomes negatively charged. When the conductor U is removed from contact with the charged conductor it is found that it remains charged, provided it is perfectly insulated from all other conductors. A conductor which is thus charged by being placed in contact with another conductor is said to be "charged by contact."*

*It is also possible to charge the surface of a dielectric by placing it in contact with a charged conductor or a charged dielectric. Very little is known concerning the exact distribution of such charges on dielectrics. Fortunately, where a charged conductor is in contact with a dielectric it is in general immaterial whether the charge is considered as "residing" on the surface of the conductor or on the surface of the dielectric or on both. These *contact* charges on dielectrics must not be confused with the *induced* charges discussed later.

It is also found by experiment that a conductor can be charged without placing it in actual contact with a charged conductor; merely placing the originally uncharged conductor U in the vicinity of a charged body C will cause the originally uncharged conductor U to become charged. In this case, however, when the originally uncharged conductor U is perfectly insulated, it is found that the portion of this conductor U nearer the charged body C receives a charge of the opposite sign to that on C while the more remote portion of the conductor U receives a charge of the same sign as that of C. In other words, the conductor U becomes charged by "induction" in a manner similar to that by which a piece of soft iron placed in a magnetic field becomes magnetised by induction. There is, however, an important difference between the phenomenon of electrostatic induction in a conductor and the phenomenon of magnetic induction in iron. When a piece of iron which is magnetised by induction is divided into two parts and the two parts are separated slightly, so that a narrow gap is formed between them, magnetic poles are found to exist on the surfaces of the iron forming the walls of this gap. When a conductor U (which may be made in two parts originally in contact) is charged by induction and the two parts U_1 and U_2 are separated a slight distance, *the surfaces of* U_1 *and* U_2 *forming the walls of this gap are not charged.* Again, when the piece of iron which is magnetised by induction is separated into two parts and either part is removed from the magnetic field, this portion of the original piece of iron is found to have equal and opposite poles which disappear almost entirely when the iron is jarred; however, when a conductor which is charged by induction is separated into two parts U_1 and U_2, it is found that the two portions of the conductor retain their charges, one part a positive charge and the other part a negative charge, as long as the two parts are kept perfectly insulated from each other and from all other conductors. The charges on the two portions of the conductor also remain unaltered in amount (but not in distribution, as we shall see later) even when these two portions are removed from the vicinity of the conductor which induced the charges and are separated from each other by any distance. Hence another important difference between the phenomenon of electrostatic induction in a conductor and the phenomenon of magnetic induction; the total strength of the poles on a mag-

netic substance is always zero, no matter into how many pieces the substance may be broken, but the positive and negative charges induced on a conductor may be separated from each other by dividing the conductor.

Experiment also shows that electric charges are induced on a dielectric which is placed in the vicinity of a charged body, in the same manner that charges are induced on a conductor. In the case of a solid or liquid dielectric separated from the charged body by air the portion of the dielectric nearer the charged body shows a charge of the opposite sign to that on the charged body, while the portion of the dielectric more remote from the charged body shows a charge of the same sign as that on the charged body. However, when a dielectric which is thus charged by induction is separated into two parts, it is found that charges *are produced* on the walls of the gap between the two parts, a negative charge on the side of the gap nearer the positive induced charge on the original surface of the dielectric, and a positive charge on the side of the gap nearer the negative charge induced on the original surface of the dielectric. Also, when the dielectric as a whole, or any portion of it, is removed from the vicinity of the charged body which induces the charges on it, it is found that these induced charges entirely disappear. In other words, the phenomenon of electrostatic induction in a dielectric is of a similar nature to the phenomenon of magnetic induction in a magnetic substance, except that it is impossible to produce by electrostatic induction a "permanently electrified" dielectric. The above facts are true only of a dielectric which is an absolute non-conductor. Every dielectric, however, is a conductor to a certain extent, and the resultant effect produced in a dielectric when it is placed in the vicinity of a charged body is therefore a combination of the effect due to its true dielectric or insulating property and the effect due to its conducting property. We shall see later how both these effects may be taken into account.

An electric charge can be induced only at the surface of separation of two dissimilar substances. There is no way of determining experimentally "on" which of the substances in contact the charge is induced; for example, when a piece of glass is placed in air in the vicinity of a charged body, there is no way of determining whether the induced charge at the surface of separation between the air and the glass is "on"

the surface of the air or "on" the surface of the glass. However, just as in the discussion of the phenomena of magnetism it was found convenient to assume that air was non-magnetic (and therefore at the surface of separation between air and any other substance the induced poles are entirely on the other substance), so in the discussion of the phenomena of electrostatics, it is found convenient to assume that at the surface of separation between air* and any other substance, there is no charge induced on the air. In general, at the surface of separation between any other two substances, whether they be dielectrics or conductors, it will then be necessary, in order to account for the observed phenomena, to assume that a charge is induced on the surface of both these substances.

130. Point-Charges. — It is found to be impossible to produce a finite electric charge at a point in space, but just as in the discussion of magnetic phenomena it was found convenient to make use of the conception of a point-pole, so in the discussion of electrostatic phenomena it is convenient to consider a charge of finite amount concentrated in a point. Such a charge may be called a *point-charge.* A physical approximation to a point-charge is the charge on a small area.

131. Properties of Electric Charges. — When all the electric charges produced in any manner whatever, *including the charges induced on dielectrics,* are taken into account, it is found that the forces produced on one another by any number of charged bodies may be accounted for by attributing to these electric charges the following properties:

1. Like charges repel each other and unlike charges attract each other.

2. When a charge of one sign is produced on any body an equal and opposite charge is produced either on the same or on some other body.

3. Two point-charges q and q' located at a distance r apart *repel* each other with a force proportional to the products of the quantities q and q' of these charges and inversely proportional to the square of the distance between them, *independent of the nature of the medium between them;* that is, with a force

*Strictly, this assumption is permissible only for one definite pressure; a pressure of 760 mm. of mercury is the adopted standard.

$$f = k\frac{qq'}{r^2} \tag{1}$$

where k is a constant depending upon the units in which q, q', r and f are measured. Since we have not as yet specified the unit in which an electric charge is to be measured, we may select this unit such that the constant k in the above equation is unity. We then have that the force of repulsion between two point-charges having the values q and q' is

$$f = \frac{qq'}{r^2} \tag{1a}$$

When q and q' are of the same sign this force is positive, and therefore there is an actual repulsion; when q and q' are of opposite signs this force is negative, that is, there is an actual attraction. This agrees with the first property stated above. The line of action of the force produced by one point-charge on another is the line drawn between them.

The *resultant* force produced on a given point-charge q' by any number of point-charges is the vector sum of all the individual forces acting on that charge, that is, is

$$F = q' \bar{\Sigma} \frac{q}{r^2} \tag{1b}$$

where the summation $\bar{\Sigma} \frac{q}{r^2}$ includes all the charges in the vicinity, induced or otherwise, on conductors or dielectrics. Since the quantity of charge induced on a dielectric depends upon the nature of the dielectric, this *resultant* force will also depend upon the nature of the surrounding dielectrics, but the force due directly to any given charge q, whether on a conductor or induced on a dielectric, is independent of the nature of the medium between q and q'.

The unit of electric charge is defined by equation (1*a*); that is, a unit point-charge is a charge which *repels with a force of one dyne an equal point-charge placed one centimeter away.* This unit is called the *c. g. s. electrostatic unit* of charge.

Since the properties of electric charges are of exactly the same form as the properties of magnetic poles, with the one exception that a charge of one sign can exist by itself on a conductor, it follows that the definitions and deductions in regard to the properties of magnetic poles can be applied directly to electric charges, except such properties as were deduced as a consequence of the

fact that equal and opposite poles always exist on the same body.*

132. Electrostatic Field of Force. — Electrostatic Intensity. — Any region of space in which an electric charge would be acted upon by a mechanical force is called an *electrostatic field of force.* The *intensity of the electrostatic field* at any point is defined as the force in dynes which would be exerted by the agents producing the field on a unit point-charge placed at that point. The names *electrostatic intensity* and *electrostatic force* are also used for this quantity. The electrostatic intensity at any point due to a point-charge q at a distance r centimeters away is then

$$H = \frac{q}{r^2} \tag{2}$$

and the total field intensity at any point P due to any number of point-charges is the vector sum

$$H = \overline{\Sigma} \frac{q}{r^2} \tag{2a}$$

where q is the value of the point-charge at any point and r is the distance in centimeters of this charge from the point P, and the summation includes all the charges, induced or otherwise, on all the conductors and dielectrics in the field.

The total field intensity at any point P due to any charged surface is the vector integral

$$H = \overline{\int_S} \frac{\sigma \, ds}{r^2} \tag{2b}$$

where ds is any elementary area of the surface, σ is the charge per unit area, or the *surface destiny* of the charge at ds, and r is the distance of ds from P.

133. Lines of Electrostatic Force. — Flux of Electrostatic Force. — Lines of electrostatic force may be drawn in the same manner as lines of magnetic force, that is, lines of electrostatic force are lines drawn in such a manner that the direction of each line at each point coincides with the direction of the electrostatic intensity at that point and the number of these lines per unit area at any point, normal to their direction, is equal to the value of the electrostatic intensity at that point. The number of

*The same symbols will as a rule be used throughout for the corresponding quantities; in any problem in which both electrostatic and magnetic quantities must be used, the former may be distinguished by the subscript "e."

these lines of electrostatic force crossing any area is defined as the *flux of electrostatic* force across that area. The mathematical expression for the flux of electrostatic force across any surface S is then

$$\psi = \int_S (H \cos \alpha)\, ds \tag{3}$$

where ds is any elementary area of this surface, H the electrostatic intensity at ds and α the angle between the direction of this electrostatic intensity and the normal to ds. Gauss's Theorem, which is simply a consequence of the inverse square law, also holds for electric charges; that is, the number of lines of electrostatic force outward across any closed surface is equal to the algebraic sum of the charges inside the surface. The mathematical expression of this fact is

$$\int_{|S|} (H \cos \alpha)\, ds = 4\,\pi \,\Sigma\, q \tag{3a}$$

where $\int_{|S|}$ represents the integral over the *closed* surface.

134. Lines of Electrisation. — We have already seen that when a charged conductor is separated into two parts by a narrow gap, there is no electric charge produced on the walls of this gap. It is also found that when a closed cavity of any kind is formed inside a charged conductor, no electric charge is produced on the walls of this cavity no matter how the external surface of the conductor is charged. A charge can be produced on the walls of such a cavity only by introducing a charged body into this cavity. Hence *in a charged conductor there is nothing analogous to lines of magnetisation inside a magnetised body.*

In the case of a dielectric charged by induction, however, we have seen that in general charges do appear on the walls of a gap cut in the dielectric, just as magnetic poles appear upon the walls of a gap cut in a magnetised body. It is possible, however, just as in the case of a magnetised body, to cut in a dielectric which is charged by induction a gap in such a direction that no charge will appear on the walls of the gap. Hence we may consider a dielectric which is charged by induction to be made up of filaments the walls of each of which have such a direction that were this filament separated from the rest of the dielectric by a gap of infinitesimal width, there would be no charges induced on the lateral walls of this filament. The two ends of such a filament in the original surface of the dielectric will then have

equal and opposite charges. We may then take each filament of such a size that it terminates at each end in a charge which has the numerical value $\frac{1}{4\pi}$. The positive sense of such a filament is taken as the direction along it from its negative to its positive end; that is, these filaments are considered as running through the dielectric from its negatively charged end to its positively charged end. These filaments are called *lines of electrisation.* The intensity of electrisation at any point in a body is defined as the charge per unit area which would appear on a gap cut in the body at this point perpendicular to the line of electrisation through this point. The direction of the intensity of electrisation at any point is taken as the direction of the line of electrisation through this point. Lines of electrisation, lines of electrostatic force, and the electric charge induced on the surface of a dielectric are then related to one another in exactly the same manner as lines of magnetisation, lines of magnetic force and the magnetic poles induced on the surface of a magnetic body. That is, the number of lines of force ending on a negative charge induced on a dielectric is equal to the number of lines of electrisation originating from that charge, and the number of lines of electrisation ending on a positive charge induced on the surface of a dielectric is equal to the number of lines of electrostatic force originating from that charge. The relation between the intensity of electrisation at any point in the surface of a dielectric and the charge induced on that surface is then

$$\sigma = J \cos \alpha \qquad (4)$$

where σ is the surface density of this induced charge at this point, J the intensity of electrisation of the dielectric at this point, and α the angle between the direction of the intensity of electrisation at this point and the direction of the normal drawn *outward* from the surface of the dielectric at this point. (See Article 46.)

The number of lines of electrisation crossing an elementary surface ds the normal to which makes an angle α with the direction of the intensity of electrisation is

$$dN = 4\pi (J \cos \alpha)\, ds \qquad (5)$$

Since we have assumed (Article 129) that no charges are induced on the surface of the air in contact with any other substance, the intensity of electrisation in air is zero.

It should be clearly kept in mind that the above relations hold only for charges *induced on dielectrics*. Since every dielectric (other than a perfect vacuum) is a conductor to a slight extent, charges are also induced on the dielectric due to its conducting property; these latter charges, however, are usually negligible in practical work.

135. Lines of Electrostatic Induction. — Flux of Electrostatic Induction. — The algebraic sum of the number of lines of electrisation and the number of lines of electrostatic force across any surface in an electrostatic field is called the number of *lines of electrostatic induction* across that surface, or the *flux of electrostatic induction* across that surface. Experiment shows that, except in the special case of certain crystals, the intensity of electrisation at any point in a dielectric is in the same direction as the intensity of the electrostatic field. Hence the mathematical expression for the number of lines of electrostatic induction crossing an elementary surface ds_n *normal* to the direction of the field intensity is

$$d\phi = (H + 4\pi J)\, ds_n$$

where H is the intensity of the electrostatic field at ds, and J the intensity of electrisation in the dielectric at ds. The number of lines of electrostatic induction per unit area at any point normal to their direction is called the *electrostatic flux density** at this point. The relation between electrostatic flux density, field intensity and intensity of electrisation is then

$$B = H + 4\pi J \qquad (6)$$

Hence the flux of electrostatic induction across any elementary surface ds may also be written

$$d\phi = (B \cos \alpha)\, ds \qquad (7)$$

where B is the electrostatic flux density at ds and α is the angle between the direction of this electrostatic flux density and the normal to ds.

Since the intensity of electrisation in air is zero, the lines of electrostatic force and the lines of electrostatic induction in air

*The electrostatic flux density divided by the factor 4π is called the electrostatic *polarisation* or the electric *displacement*. The factor 4π comes in from the fact that one line of polarisation or one line of electric displacemen is considered as originating from each unit charge on a conductor, while from a unit charge on a conductor there originate 4π lines of electrostatic *induction*.

are identical, and therefore the electrostatic intensity and the electrostatic flux density in air are equal.

136. Dielectric Constant. — Experiment shows that the electrostatic flux density produced in any dielectric when placed in an electrostatic field is directly proportional to the resultant field intensity, and except in the special case of certain kinds of crystals, the direction of the electrostatic flux density coincides with the direction of the field intensity. The ratio of the electrostatic flux density to the field intensity is called the dielectric constant of the dielectric. That is, calling B the electrostatic flux density at any point in the dielectric, and H the resultant electrostatic intensity at this point, the dielectric constant of the dielectric at this point is

$$K = \frac{B}{H} \tag{8}$$

The dielectric constant is exactly analogous to magnetic permeability. The values of the dielectric constants for a few substances are

Solid paraffin	2.29
Paraffin oil	1.92
Ebonite	3.15
Mica	6.64
Glass	6.5 to 7.5
Distilled water	76
Alcohol	26
Ordinary gases	1.000
Perfect vacuum	0.9995

Combining equations (6) and (8) we have that $H + 4\pi J = KH$ and therefore that

$$J = \frac{(K-1)\ H}{4\pi} \tag{9}$$

Hence in any dielectric for which the dielectric constant is unity the intensity of electrisation is zero, and therefore there are no lines of electrisation, and hence there are no charges induced on the surface of such a dielectric. In a dielectric for which the dielectric constant is greater than unity, the intensity of electrisation is not zero but a positive quantity, and therefore the field intensity is less than the flux density; hence the number of lines of electrostatic force inside a dielectric of which the dielectric

constant is greater than unity is always less than the number of lines of electrostatic induction.

137. A Closed Hollow Conductor an Electrostatic Screen. — It is found by experiment that an electric charge cannot be produced on the *inside* surface of a hollow closed conductor by charging the conductor either by contact with, or by induction from, any agent whatever *outside* the conductor; nor can an electrostatic field be produced *inside* the space enclosed by the conductor by any agent whatever *exterior* to the conductor, provided in each case there is no electric current in the conductor. These two important facts were discovered by Faraday, who built a large insulated metal chamber in which he set up the most delicate measuring instruments he could devise; he was unable to detect the slightest electrostatic effect inside this chamber no matter how highly the chamber was electrified on the outside. All deductions from this discovery of Faraday's have been found to be in accord with experiment, and we therefore accept as a fundamental law of electrostatics that any region completely enclosed by a conductor is absolutely protected from external electrostatic effects. It is also found that a metal gauze or cage forms a practically perfect screen.

138. The Resultant Electrostatic Intensity Within the Substance of a Conductor in which there is no Electric Current is Always Zero. — An important deduction from Faraday's discovery is that the resultant electrostatic intensity within the substance of a charged conductor is zero, provided there is no current in the conductor, no matter how the conductor may be charged. For, should a closed cavity be made in this conductor, no charge would appear on the walls of this cavity, and therefore the original electrostatic field would not be altered; therefore the electrostatic field inside this cavity must be the same as originally existed in the conducting material which filled this cavity, since the electrostatic intensity at any point due to any given distribution of electric charges is independent of the material at the point in question. But the electrostatic intensity in the cavity is zero, and since the charges producing the field are not altered by forming the cavity, the electrostatic intensity in the conducting material which originally filled this cavity must also have been zero.

Since the resultant intensity in a conductor is zero, it follows

that when a conductor is placed in an electrostatic field, the induced charges which appear must be distributed in such a manner that the field within the substance of the conductor due to these induced charges is just equal and opposite to the original field in the space occupied by the conductor.

139. The Total Charge Within any Region Completely Enclosed by a Conductor is Always Zero. — We have just seen that the electrostatic intensity is zero within the substance of a conductor in which there is no electric current. Hence when any region of space is completely enclosed by such a conductor the total flux of electrostatic intensity out through any surface drawn within the substance of this conducting shell is zero, since there are no lines of electrostatic force crossing this surface. Hence, by Gauss's Theorem, the total charge *inside* such a surface is zero. Also, since it is one of the fundamental properties of electric charges that a charge of one sign cannot be produced without at the same time producing an equal charge of the opposite sign, the total charge *outside* such a closed surface must also be zero. Hence a closed conducting surface divides all space into two parts in each of which parts the total charge is zero. For example, let A be an insulated conductor which has

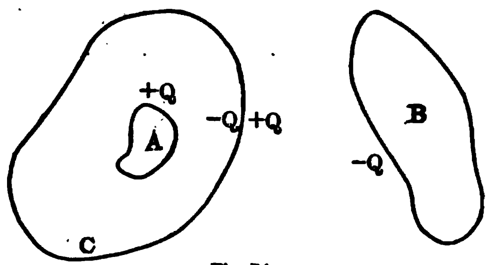


Fig. 74.

a charge of Q units; then on some other conductor or conductors B there must be an equal and opposite charge of $-Q$ units. Let A be completely surrounded by an insulated conducting shell C; there must then be induced on the inside of this shell a charge of $-Q$ units and on the outside of this shell a charge of $+Q$ units.

140. The Electrostatic Intensity Just Outside a Charged Conductor is Normal to the Surface of the Conductor. —By the same process of reasoning as that employed in Article 50, by which it was shown that the tangential components of the magnetic field intensities just inside and just outside a magnetically charged

surface at any point are equal, it can be shown that the tangential components of the electrostatic intensities just inside and just outside an electrically charged surface are also equal at each point of the surface. In the case of a charged conductor in which there is no electric current we have just seen that the resultant electrostatic intensity inside the surface is zero, and therefore inside the surface the tangential component is zero. Hence the tangential component of electrostatic intensity just *outside* the surface must also be zero. (When there is an electric current in the conductor the tangential component of the electrostatic intensity at the surface of the conductor is not zero, but is equal to $\frac{\rho\sigma}{3\times10^{10}}$ where ρ is the specific resistance of the conductor in ohms per centimeter cube and σ is the current density in abamperes per square centimeter, see Articles 102 and 148; this quantity, however, is usually negligibly small.) Hence the important result: *the resultant electrostatic intensity just outside the surface of a charged conductor is normal to the surface of the conductor;* or, stated in other words, the resultant lines of electrostatic force are always perpendicular to the surface of a conductor where they enter or leave it. This relation is absolutely true in case there is no electric current in the conductor and also holds to a close degree of approximation in any practical case when there is a current in the conductor.

141. The Electrostatic Flux Density Just Outside a Charged Conductor is Independent of the Nature of the Surrounding Dielectric. — Let the dielectric in contact with the conductor have a dielectric constant K, and let the surface density of the charge on the conductor at any point P be σ. A charge will also be *induced* on the surface of the dielectric in contact with the conductor at P; let σ' be the value of this induced charge. Draw around P a small right cylinder with its axis normal to the surface of the conductor at P and one end of the cylinder inside the surface and the other outside the surface. Let H be resultant electrostatic intensity at P just outside the surface of the conductor, and let the end of the cylinder have the area ds. Then when the ends of the cylinder are infinitely close to

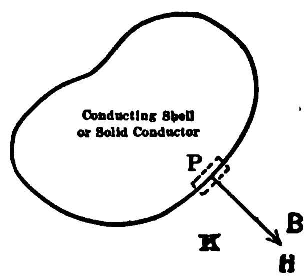


Fig. 75.

the surface, the electrostatic intensity outside the conductor will be normal to ds. Hence the total flux of electrostatic force across the outside end of the cylinder will be Hds. Across the lateral walls of the cylinder outside the conducting surface the flux of force will be zero, since these lateral walls are parallel to the direction of the field intensity. Across the walls of the portion of the cylinder inside the surface the flux of force will likewise be zero, since there is no electrostatic field inside the conductor. Hence the total flux of force outward through the walls of this cylinder is Hds. By Gauss's Theorem this must be equal to 4π times the total charge inside this cylinder, that is, equal to $4\pi(\sigma+\sigma')\,ds$. Hence

$$H=4\pi(\sigma+\sigma')$$

But since the lines of electrisation (which coincide in direction with the lines of force) are perpendicular to the surface at P and are *into* the dielectric from its surface, the intensity of electrisation of the dielectric just outside the conductor at P is

$$J=-\sigma'$$

see equation (4). From equation (6) the electrostatic flux density at P is $B=H+4\pi J$. Therefore substituting for H and J their values in terms of σ and σ', we have

$$B=4\pi(\sigma+\sigma')-4\pi\sigma'=4\pi\sigma$$

That is, the electrostatic flux density at any point just outside a conductor is

$$B=4\pi\sigma \qquad (10)$$

where σ is the surface density of the charge *on the conductor* at this point, *and is independent of the nature of the dielectric in contact with the conductor at this point.** The resultant electrostatic intensity just outside a charged conductor, on the other hand, does depend upon the nature of the dielectric in contact with the conductor, and is equal to

$$H=\frac{4\pi\sigma}{K} \qquad (11)$$

where K is the dielectric constant of the dielectric in contact with the conductor at the point in question.

An important difference between lines of magnetic induction

*In the deduction of this relation the *contact* charge, if any, produced on the surface of the dielectric, is neglected. In case there is a contact charge on the dielectric this same relation holds when σ is taken to represent the surface density of the charge on the conductor plus the *contact* charge on the dielectric.

and lines of electrostatic induction should be noted in this connection. Lines of magnetic induction are always closed curves. On the other hand, lines of electrostatic induction which are due entirely to electric charges originate from and end on conductors (or contact charges on dielectrics); 4π of them originate from each unit positive charge on a conductor and 4π of them end on each unit negative charge on a conductor. They do not, however, end on charges *induced* on dielectrics, but pass *through* these charges (see Article 54), in general making an abrupt change in direction at the surface of a dielectric on which a charge is induced. Lines of electrostatic force, however, are strictly analogous to lines of magnetic force; 4π of them originate from every unit positive charge and 4π of them end on every unit negative charge, whether these charges be on a conductor or are induced charges on a dielectric. Hence in any dielectric which has a dielectric constant greater than unity there are fewer lines of force than lines of induction, for the lines of induction all pass through the dielectric while some of the lines of force end on its surface.

142. Conditions which Must be Satisfied at any Surface in an Electrostatic Field. — By identically the same process of reasoning as that employed in Articles 49 and 50 it can be shown that at every surface in an electrostatic field other than a conducting surface the following conditions must be satisfied:

1. The normal components of the electrostatic induction on the two sides of any surface at any given point in this surface must be equal.

2. The tangential components of the electrostatic intensity on the two sides of any surface at any given point in this surface must be equal.

At the surface of a conductor the surface conditions are, as we have just seen, that

1. Both the electrostatic intensity and the electrostatic induction inside the surface are zero.

2. Both the electrostatic intensity and the electrostatic induction just outside the surface are normal to the surface.

From these surface conditions the distribution of the charges on conductors and those induced on dielectrics may be calculated in certain simple cases.

143. Electrostatic Potential. — The work that would be done

by the agents producing an electrostatic field *on* a unit positive point-charge, were such a point-charge moved from any point P in the field to infinity, is called the *electrostatic potential* of the field at the point P. The electrostatic potential at any point P due to a point-charge q at a distance r from P is then

$$V = \frac{q}{r} \tag{12}$$

and is independent of the path over which the charge is moved from P to infinity, and therefore depends only upon the relative position of the charge and the point P. (See Article 59.) Since electrostatic potential is the ratio of work to charge, and since both work and charge are scalar quantities, electrostatic potential is also a scalar quantity. Hence the resultant electrostatic potential at any point due to any number of electric charges is the algebraic sum of the potentials due to each charge separately. Therefore the electrostatic potential at any point P due to a charged surface of any kind is

$$V = \int_S \frac{\sigma\, ds}{r} \tag{12a}$$

where ds represents any elementary area of the surface, σ the surface density of the charge at ds, and r the distance of the point P from ds.

144. Difference of Electrostatic Potential. — The difference of electrostatic potential between any two points 1 and 2, or specifically, the *drop* in electrostatic potential from the point 1 to the point 2, is the work that would be done *by* the agents producing the electrostatic field *on* a unit positive point-charge were such a charge moved from the point 1 to the point 2. Calling dl an elementary length in any path from the point 1 to the point 2, H the electrostatic intensity at dl, and θ the angle between the direction of the electrostatic intensity H and the direction of the length dl, the drop of electrostatic potential from 1 to 2 along this path is then

$$V_{12} = \int_1^2 (H \cos \theta)\, dl \tag{13}$$

When there is no contact or induced electromotive force in the path from 1 to 2, the drop of electrostatic potential from any point 1 to any point 2 is independent of the path over which the charge

is moved, and therefore under this condition the drop of electrostatic potential around any closed path in the field is zero.

However, as we shall see presently, when there is any contact or induced electromotive force in the field, this contact or induced electromotive force also produces an electrostatic field the *lines of force representing which are closed loops* passing through the surface of contact in the case of a contact electromotive force and linking the lines of magnetic induction in the case of an induced electromotive force. In such a case the drop of electrostatic potential around any closed loop is equal to the algebraic sum of the contact and induced electromotive forces in the loop, provided the drop of electrostatic potential and the electromotive forces are expressed in the same unit. (See Article 148.) Hence, when there are any contact or induced electromotive forces in an electrostatic field the difference of electrostatic potential between any two points depends not only upon the position of the two points but also upon the path over which the unit charge is moved. For example, in the conductor forming the armature winding of a continuous current generator the resultant electrostatic intensity is zero when there is no current in the armature conductors (see Article 138). Hence the drop of electrostatic potential *through the winding* from the positive to the negative terminal is zero, but the drop of electrostatic potential from the positive to the negative terminal *through the surrounding air* is equal to the electromotive force of the generator. Compare also with the relation between drop of magnetic potential around a closed loop and the magnetomotive force linked by this loop, Article 113.

145. Electrostatic Equipotential Surfaces. — An electrostatic equipotential surface is a surface drawn in an electrostatic field in such a manner that the drop of electrostatic potential along any path in the surface is zero. Such a surface will intersect the lines of electrostatic force at right angles. For, were any line of force not perpendicular to the surface where it crosses the surface, the electrostatic intensity at this point of intersection would have a component parallel to the surface, and therefore work would be done on a unit point-charge were the latter moved along the surface in the direction of this component; but this is contrary to the condition that the surface is drawn in such a manner that there is no difference of potential between any two points in its surface.

When a conductor in which there is no electromotive force is placed in the electrostatic field, the conductor must become an equipotential surface, for we have already seen that the electrostatic intensity is always normal to the surface of a conductor in which there is no electric current, and therefore the drop of potential along any path in the surface of the conductor is zero.

146. Parallel Plate Electrometer. — The following example will illustrate the principles stated above, and will at the same time show how electric charge and difference of electrostatic potential may be measured. By the same process of reasoning as that employed in Article 37 it can be shown that the electrostatic intensity due to a uniformly charged plane surface is in the direction normal to this surface at any point at a perpendicular distance from the surface small compared with the distance of the point from the perimeter of the surface, and that its value is

$$H = 2\,\pi\,\sigma_n \tag{14}$$

where σ_n is the surface density of the net charge at this surface. By net charge is meant the algebraic sum of the charge on the conductor, the contact charge (if any) on the dielectric in contact with the conductor, and the charge induced on the dielectric. Consider two equal plane metal discs A and B placed parallel and opposite to each other and immersed in a dielectric of which the dielectric constant is K, and let this dielectric completely fill all the surrounding space.

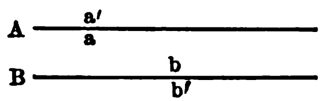


Fig. 76.

Let the plates A and B be given charges of Q and $-Q$ units respectively. The only charges induced on the dielectric will then be those at the surface of separation between the metal plates and the dielectric. The charges Q and $-Q$ and the charges induced on the dielectric must be distributed in such a manner that (1) the electrostatic intensity at any point inside the plates is zero, and (2) the electrostatic intensity at each point in the dielectric infinitely close to the surface of the plates is normal to the surface at this point. A uniform distribution of the net charge at the two surfaces a and b which face each other and no charge on the "back" surfaces a' and b' will satisfy these conditions for all points except those near the edges of the plates. For, calling σ_n and $-\sigma_n$ the surface densities of the net charge at the surfaces a and b respectively, the resultant intensity (from equation 14) *inside* each plate will be

$2\pi\sigma_n - 2\pi\sigma_n = 0$, and the intensity at any point in the space between the plates will be $2\pi\sigma_n + 2\pi\sigma_n = 4\pi\sigma_n$ and will be perpendicular to these surfaces, while at each point infinitely close to the surfaces a' and b' the intensity will be $2\pi\sigma_n - 2\pi\sigma_n = 0$, which relations satisfy the surface conditions 1 and 2. It can also be shown in that there is only one possible distribution of charges which can satisfy these surface conditions; hence at all points at a considerable distance from the edges of the plates the charges at the surfaces of separation between the dielectric and the plates must be distributed uniformly over the surfaces a and b and there can be no charge on the surfaces a' and b' (except near their edges).

Let σ be the value of the total charge on the conductor and the *contact* charge, if any, on the dielectric at each point of the surface a and σ' the value of the charge *induced* on the dielectric at the surface a. Then $\sigma_n = \sigma + \sigma' = \sigma \dot{-} J$, where J is the intensity of electrisation of the dielectric in the direction from a to b. But $J = \frac{(K-1)H}{4\pi}$, equation (9*a*), whence the resultant field intensity at any point in the space between the plates, not too close to their edges, is

$$H = \frac{4\pi\sigma}{K}$$

where σ *is the surface density of the charge on the conductor.* (Compare with equation 11.) This electrostatic intensity is therefore perpendicular to the surfaces of the plates, and the electrostatic field between the two plates is a uniform field except for points not too close to the edges of the plates.

The force of attraction between the two plates may be readily calculated. For, of the resultant intensity $H = \frac{4\pi\sigma}{K}$, half must be due to each plate, that is the resultant field intensity due to each plate is $\frac{2\pi\sigma}{K}$. This, then, is the value of the force due to the plate A, say, on each unit charge on the plate B (or, *vice versa*, the force due to the plate B on each unit charge on the plate A). Hence the pull on each unit area of the plate A is $\frac{2\pi\sigma^2}{K}$, and when the linear dimensions of the plates are equal and are large compared

to their distance apart, so that the non-uniformity of the distribution of the charge at their edges may be neglected, the total force of attraction between the two plates is

$$F=\frac{2\,\pi\,\sigma^2\,S}{K}=\frac{2\,\pi\,Q^2}{K\,S}$$

and therefore

$$Q=\sqrt{\frac{KSF}{2\,\pi}} \qquad (15a)$$

where S is the area of each plate and Q the total charge on each plate. The force F may be readily measured by suspending one of the plates from one arm of a suitable chemical balance.

When the plates are separated by air, the force of attraction between the two plates is $F=\frac{2\,\pi\,Q^2}{S}$, since the dielectric constant of air is unity. In this case the charge on each plate is numerically equal to

$$Q=\sqrt{\frac{S\,F}{2\,\pi}} \qquad (15b)$$

Hence, by measuring the force of attraction between the two plates the charge on each p ate may be calculated. Again, from the fact that the electrostatic field between the two plates is uniform, the difference of electrostatic potential between the plates may also be calculated. For, calling D the distance between the two plates, we have, from equation (13), that this difference of electrostatic potential is

$$V=\int_0^D H\,dl=\frac{4\,\pi\,\sigma\,D}{K}=\frac{4\,\pi\,Q\,D}{K\,S}=D\sqrt{\frac{8\,\pi\,F}{K\,S}} \qquad (16)$$

When the plates are separated by air, the dielectric constant is unity, and therefore the difference of electrostatic potential between the two plates is

$$V=D\sqrt{\frac{8\,\pi\,F}{S}} \qquad (16a)$$

The arrangement of parallel plates above described, which arrangement is called a *parallel plate electrometer*, gives us a means of measuring both electrostatic charge and electrostatic potential difference. The above calculations are all based upon the assumption that the non-uniformity of the charge near the edges of the plates may be neglected. The condition of uni-

formity of charge on the suspended plate and uniformity of the electrostatic field acting on it may be more closely realized by suspending only the central portion of the upper plate, and letting the rest of this plate form a flat metal ring which is kept stationary but connected by a conductor to the suspended portion. That is, the upper plate is made in two portions; a central disc suspended from the arm of the balance and an outside "guard ring" with its lower surface flush with the lower surface of the ring and separated from the disc by a narrow air gap, the ring and plate being conductively connected through the metal wires from which the disc is suspended. The surface S in the above formulas is then the area of the surface of this disc, and the charge Q is the charge on this disc.

147. Relation between Electrostatic Charge and Quantity of Electricity. — Consider an electrometer of the form described in the preceding article, and for simplicity let the plates of the electrometer be of the same area and let there be no guard ring. When the two plates are connected by a wire, any charge which

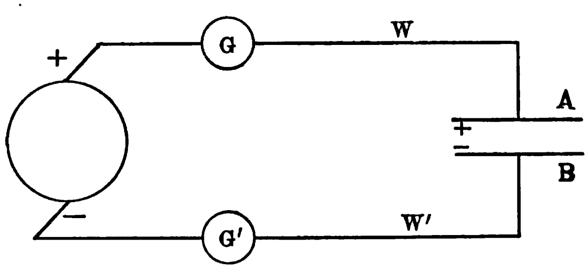


Fig. 77.

might have been on them disappears, and there is no appreciable force of attraction between them. The plates are then uncharged and are at the same electrostatic potential and also at the same electric potential, as defined in Article 87, since there is no current in the wire connecting them. (If the plates are originally charged before they are connected by the wire a momentary current will be established in the wire, as we shall see presently, but this current lasts only for a small fraction of a second.) Arrange two insulated wires W and W' in series respectively with the ballistic galvanometers G and G', and connect one end of these wires respectively to the positive and negative terminals of a generator or other source of electromotive force. Remove

the wire connecting the two plates of the electrometer and connect the other ends of the wires W and W' to the plates A and B respectively. The following phenomena will be observed:

1. A momentary electric current is established in the two galvanometers, as shown by their deflections, and therefore a certain quantity of electricity as defined in Article 80 is transferred across each section of the wire forming the galvanometer windings and the wires W and W'. If the galvanometers have been calibrated (see Article 109) the quantity of electricity transferred through each can be measured and the direction of its transfer can be observed. It will be found that equal quantities of electricity are transferred through each galvanometer (provided the connecting wires are small) and that the direction of transfer or "flow" of electricity is from the plate B which is connected to the negative terminal of the generator and towards the plate A which is connected to the positive terminal of the generator.

2. The plates A and B will become charged, as shown by their mutual attraction, the plate A positively and the plate B negatively. The charges given the two plates will also be numerically equal, as can be tested by observing the force which each produces on some other charged conductor when placed successively in the same relative position with respect to the latter.

When the value of the impressed electromotive force, or the distance apart of the plates, or the size of the plates, or the dielectric between them, is altered, it is found that the charge given each plate also changes; the quantity of electricity transferred through each galvanometer likewise changes. In each case, however, it is found that the charge given each plate is directly proportional to the quantity of electricity transferred through each galvanometer. When the charge given each plate is measured in electrostatic units, as defined in Article 131, and the quantity of electricity transferred through each galvanometer is measured in electromagnetic units as defined in Article 80, it is found that the numerical value of the charge given each plate is equal (approximately) to 3×10^{10} times the numerical value of the quantity of electricity transferred through each galvanometer. This figure 3×10^{10} is approximately equal to the velocity of light in air. Consequently, if as the unit of charge is taken a unit 3×10^{10} times the size of the electrostatic unit as defined in Article 131, the numerical value of the electric

charge given each plate will be equal to the number of electromagnet units of quantity of electricity transferred to it; similarly, if as the unit of charge is taken a unit 3×10^9 times the size of an electrostatic unit the numerical value of the electric charge given each plate will be equal to the number of coulombs transferred to it. Electric charge and quantity of electricity may then be measured in the same units; these units are related as follows:

1 *c. g. s.* electromagnetic unit or abcoulomb
$= 3 \times 10^{10}$ *c. g. s.* electrostatic units
1 coulomb $= 3 \times 10^9$ *c. g. s.* electrostatic units

(The figure 3.00×10^{10} is about as accurate as can be determined by a direct comparison of electromagnetic and electrostatic units. The electromagnetic theory of light requires that ratio between the electromagnetic and electrostatic units be exactly equal to the velocity of light in air, and the velocity of light by direct optical methods is found to be 2.998×10^{10} centimeters per second. The experimental fact that the ratio of the two kinds of units is, as closely as can be measured, equal to the velocity of light in air, is strong evidence for our belief in the truth of the electromagnetic theory.)

The phenomena just described are reversible; that is, when the charge on a conductor disappears a current is established in the conductor which transfers from each element of the surface of the conductor a quantity of electricity equal to the charge originally on this surface. For example, when the two plates of the electrometer described above are disconnected from the generator and then connected to each other through a ballistic galvanometer, it is found that the quantity of electricity transferred through the galvanometer is exactly equal to the quantity of electricity originally transferred through the galvanometers when the plates were charged (provided the plates are otherwise perfectly insulated), and the transfer of charge is in the direction from the positive to the negatively charged plate; the plates also no longer attract each other, *i.e.*, the charges on the two plates disappear.

148. Relation between Electrostatic Potential and Electric Potential. — The drop of *electrostatic* potential between any two points has been defined as the work done by the agents producing the electrostatic field when a *unit positive point-charge is*

moved from one point to the other, and the drop of *electric* potential between any two points as the work done by a continuous current in an insulated conductor connecting the two points *per unit quantity of electricity transferred across each section of the conductor.* Hence, when as the unit of charge is taken the electromagnetic unit as above defined and the erg is taken as the unit of work, the unit of electrostatic potential drop is the number of ergs of work done when one electromagnetic unit of charge is moved from one point to the other; this unit is called the electromagnetic unit of electrostatic potential difference. Similarly, when the electrostatic unit is taken as the unit of quantity of electricity and the erg as the unit of work, the unit of electric potential drop is the number of ergs of work done when one electrostatic unit of quantity is transferred across each section of the conductor; this unit is called the electrostatic unit of electric potential difference. The relations between these various units of potential difference are given in Article 87.

In general, every source of electromotive force, either contact or induced, produces an electrostatic field and electric charges appear on any conductors in the vicinity. The results of all known experiments show that *the drop of electrostatic potential through the dielectric between any two points which are connected by conductors is exactly equal to the drop of electric potential through the conductors between these points, provided both potential drops are expressed in the same units.* For example, in the experiment considered above, when the momentary currents in the galvanometers cease, the plate connected to the positive terminal of the generator must be at the same electric potential as this terminal (this follows from Ohm's Law); similarly, the plate connected to the negative terminal must be at the same electric potential as this latter terminal. Hence the difference of electric potential between the two plates is equal to the electromotive force of the generator, which electromotive force is readily measured by any of the methods described in Chapter V. The difference of electrostatic potential between the two plates can be determined by the method given in Article 146. When the values of the potential drops obtained from these two measurements are expressed in the same units they are found to be equal.

When the drop of potential through the conductor is due solely to its resistance, *i.e.*, when there is no induced or contact electro-

motive force in the conductor, we have seen (Article 102) that the product of the specific resistance of the conductor by the current density at any point gives the drop of electric potential per unit length at this point along the stream lines of the electric current, which drop of potential per unit length was defined as the electric intensity at the point in question. Since the electrostatic intensity at any point is equal to the drop of electrostatic potential per unit length measured along the line of force through this point (this follows directly from the definition of electrostatic potential, Article 144), it follows that the electric intensity and the electrostatic intensity are the same, provided they are both expressed in the same units. Since the specific resistance of ordinary insulators is very high the current density in them is very small for such potential differences as are used in practice.

149. Displacement Current. — When the two plates of the electrometer are charged by connecting them to the two terminals of a generator or other source of electromotive force, we have seen that a momentary electric current is established through the wires connecting the plates to the terminals, and that this current is in the direction away from the plate which becomes negatively charged and towards the plate which becomes positively charged. In the case of an electric current which does not vary with time we have seen that the path of the current is a continuous line which forms a closed loop lying wholly within the conductors forming the circuit. The question arises whether the path of the variable current which results in the establishment of an electric charge, such as that produced on the plates of the electrometer, is not also closed on itself; that is, *whether a momentary current is not also established through the dielectric between the plates.* To determine this, it is necessary to see whether a magnetic field due to some external source produces a force on the dielectric while the variable current exists in the conducting part of the circuit. (See definition of the measure of an electric current, Article 66.) To determine this by direct experiment is extremely difficult, though it has been done. In addition, the results of all known experiments are in accord with the assumption that an electric current always exists in the dielectric surrounding a conductor when a *variable* current exists in the conductor; that is, *when the charge on the conductor is varying* there is a current in the dielectric continuous with the current in the

conductor. There are two important differences between the properties of an electric current which can exist in a perfect dielectric and the properties of a current in a conductor. (1) The current in a conductor may be "continuous," that is, not varying with time, or it may be variable; in a perfect dielectric, however, only a *variable* current can exist. (2) The current in a perfect dielectric does not develop heat energy, while in a conductor heat energy is always produced. In an imperfect insulator both kinds of currents may exist; that part of the total current which develops heat energy due to the resistance of the dielectric is called the *conduction* current, while that part which depends upon the time variation of the electrostatic field, and therefore upon the time variation of the charges on the conductors in contact with the dielectric, is called the *displacement* current.

In the case of a conductor in contact with a perfect insulator, the charge given any element of the surface of the conductor in any small interval of time dt is equal to the quantity of electricity $dq = idt$ transferred to that surface by the current i in the conductor provided charge and current are measured in the same system of units; hence, since in this case the only current leaving that surface is the displacement current in the dielectric, it follows that the displacement current leaving that surface is

$$i = \frac{dq}{dt} \tag{17}$$

That is, *the displacement current leaving the surface of a conductor in contact with a perfect dielectric is equal to the time rate of change of the charge on that surface.* When the dielectric in contact with the conductor is not a perfect insulator, part of the conduction current coming up through the conductor to its surface leaves that surface as a conduction current through the dielectric. Hence calling i_d the displacement current in the dielectric and i_c the conduction current in the dielectric, we have that the total current coming up to the surface at any instant is

$$i = i_d + i_c \tag{18}$$

that is, the total current coming up to a surface at any instant is equal to the total current leaving that surface at that instant. Since $i_d = \frac{dq}{dt}$, where $\frac{dq}{dt}$ is the time rate of change of the charge on the surface, we have that

$$dq = (i - i_c)\ dt$$

whence
$$q = \int_0^t (i - i_c)\ dt \tag{19}$$

where q is the total charge given to the surface in any interval of time t. That is, *the total charge given the surface of a conductor in any interval of time is equal to the algebraic sum of the quantities of electricity transferred to that surface in this interval by the conduction currents flowing toward this surface.*

Since the electrostatic flux density at a point just outside a charged conductor is $B = 4\,\pi\,\sigma$, where σ is the surface density of the charge on the conductor, it follows that the current density of the displacement current just outside a charge conductor, which must be equal to the rate of change of the surface density of the charge on the conductor, is

$$\frac{d\sigma}{dt} = \frac{1}{4\pi}\frac{dB}{dt} \tag{20}$$

The results of all known experiments show that $\frac{1}{4\pi}\frac{dB}{dt}$ is a perfectly general expression for the current density of the displacement current *at any point* in a dielectric, when B represents the electrostatic flux density at that point. The quantity $\frac{B}{4\pi}$ is sometimes called the *electric displacement;* it represents the quantity of electricity transferred by the displacement current, or *displaced*, across unit area perpendicular to the direction of the electrostatic flux density.

The reader may find it difficult at first to grasp the idea of displacement current in an insulator, but a re-reading of Article 64 will make evident how such a variable current *might* occur in a dielectric. If electricity has the property of an incompressible fluid filling all space, then what is called a positive charge on a conductor represents a displacement of electricity in one direction across the surface of separation between a conductor and a dielectric, and what is called a negative charge on a conductor represents a displacement of electricity across this surface in the opposite direction. A corresponding displacement of electricity occurs in the dielectric and also in the conductor, but in the case of a dielectric the amount of displacement is limited by a property of the dielectric analogous to the elastic property of a solid, such as that forming the walls of the cellular structure

described in Article 64, while a conductor has no such elastic property. What is called a charged surface is then analogous to the surface of separation between two bodies which are differently strained.

150. Dielectric Strength. — Discharge from Points. — Corona. — Since the quantity of electricity displaced across unit area in a dielectric is proportional to the electrostatic flux density at this area, which in turn is proportional to the electrostatic intensity at this point, the strain produced in the dielectric at any point may be looked upon as proportional to the field intensity at that point. It is found by experiment that when the electrostatic intensity at any point in a dielectric is increased up to a certain definite value, the dielectric is ruptured at this point, as is evidenced by the spark which occurs, which in general extends through the dielectric from one of the charged conductors to the other. This is analogous to rupture of any substance when it is mechanically strained up to its breaking point. The value of the electrostatic intensity which causes the rupture of a dielectric is called the *dielectric strength* of the dielectric.

Recent experiments show that the dielectric strength of ordinary insulators, such as rubber, mica, porcelain, cloth, etc., depends not only upon the chemical nature of the insulator but also upon its thickness. The dielectric strength of a thin piece of insulation is greater than the dielectric strength of a thick piece; also, several thin layers making a given thickness have a greater dielectric strength than a single layer of the same total thickness. In engineering work dielectric strength is usually expressed as so many volts per inch; see Article 102.

Since the electrostatic intensity just outside a conductor is proportional to the surface density of the charge on the conductor (see equation 11) and since the surface density in turn is in general greatest at sharp points in the surface of a conductor, the dielectric as a rule breaks down first at the sharpest point of the conductor. The protective effect of a lightning rod is based upon this fact.

In the case of a non-uniform electrostatic field, such as the field about the wires of a transmission line or in the insulation of a cable, the field intensity under certain conditions can exceed the dielectric strength of the insulation in the immediate vicinity of the wire without exceeding the dielectric strength in

the rest of the insulation. Under these conditions the part of the insulation where the dielectric strength is exceeded apparently becomes a fairly good conductor. In the case of gaseous and liquid insulators this change in the nature of the insulation is accompanied by the appearance of a bluish light in the vicinity of the wire. The region around the wire in which this change takes place is called the "corona," and the whole phenomenon is called the "corona effect."

151. Electric Capacity. — Electric Condenser. — When the electrostatic potential drop along the lines of electrostatic induction varies with time, there is also a variation of the electrostatic flux density, and, as noted in Article 149, this variation of the electrostatic induction with respect to time gives rise to an electric current (*i.e.*, the displacement current). The numerical value of the ratio of the displacement current through a given portion of a dielectric between any two equipotential surfaces to the time rate of change of the electrostatic potential difference between these two surfaces is defined as the *electric capacity* of this portion of the dielectric. That is, when the displacement current in the given portion of the dielectric is i and the time rate of change of the electrostatic potential difference is $\frac{dv}{dt}$, the capacity of the given portion of the dielectric is

$$C=\frac{i}{\frac{dv}{dt}} \tag{21}$$

Compare with the definition of self inductance of a circuit, Article 116. When the current is expressed in abamperes, the potential difference in abvolts and the time in seconds, the unit of capacity is called the *abfarad;* when these quantities are expressed in amperes, volts and seconds respectively the unit of capacity is the *farad.* When the current is expressed in *c. g. s.* electrostatic units (*i.e.*, electrostatic units of quantity per second), the potential difference in electrostatic units and the time in seconds, the capacity is said to be so many electrostatic units. Capacities ordinarily dealt with are of the order of one-millionth of a farad; hence the micro-farad, equal to one-millionth of a farad, is also used as a unit of capacity. The units are related to one another as follows:

1 abfarad $=10^9$ farads

1 abfarad $=10^{15}$ microfarads

1 abfarad $=9\times 10^{20}$ *c. g. s.* electrostatic units

When the equipotential surfaces between which the displacement current flows are the surfaces of conductors, the dielectric and the conductors forming these surfaces are said to form an *electric condenser*. The capacity of a condenser formed by two conducting surfaces and a dielectric may be expressed in terms of the charge on either conductor and the difference of potential between them. When the conducting surfaces are such that the displacement current leaving the first surface is the same as that entering the second, the charges on the two surfaces must be equal and opposite, see equation (19), and therefore the displacement current is $\frac{dq}{dt}$, where q is the numerical value of the charge at any instant on either conductor, see equation (17). Hence the capacity of the condenser formed by the dielectric and the two conductors is

$$C=\frac{i}{\frac{dv}{dt}}=\frac{dq}{dv} \tag{22}$$

Therefore the capacity of the condenser is equal to the *numerical value of the charge on either surface per unit difference of potential between the two.*

The capacity of such a condenser is constant, *i.e.*, $\frac{dq}{dv}$ is constant, and equal to the ratio of the charge q at any instant to the potential difference v at this instant, provided the dielectric constants of every body in the field are constant. This may be proved as follows: The electrostatic intensity H produced at any point P when a conductor is charged with q_1 units of electricity depends in general upon the size and shape of the conductor, the nature of the surrounding bodies, and the size and shape of these bodies and their position with respect to the charged conductor. For these conditions fixed, the field intensity due to the charge q_1 and whatever other charges it may induce is proportional to this charge q_1, and therefore the drop of potential due to this charge q_1 between any two points in the field is proportional to q_1, that is

$$v_1=A_1q_1$$

where A_1 is a constant depending on the size and shape of the charged conductor, the size, shape and position of any other conductors in the vicinity, and the nature, size, shape and position of the surrounding dielectric bodies, but is independent of the value of q_1. Similarly the drop of potential between these same two points due to any other charge q_2 is proportional to q_2, that is

$$v_2 = A_2 q_2$$

Hence the total drop of potential between 1 and 2 due to both charges is

$$v = v_1 + v_2 = A_1 q_1 + A_2 q_2$$

By definition, however, in the case of a condenser $q_2 = -q_1$, and therefore the difference of potential between the two conductors may be written

$$v = v_1 - v_2 = A_1 q_2 - A_2 q_2 = (A_1 - A_2)\, q_1$$

Whence, from (22), the capacity of the condenser is a constant equal to $\dfrac{1}{A_1 - A_2}$.

152. Simple Forms of Condensers. — a. Parallel Plate Condenser. — The parallel plate electrometer described in Article 146 is one of the simplest forms of electric condensers. Its capacity is calculated directly from equations (16) which give the relation between the charge on each plate and the difference of potential between them. That is, the capacity of such a parallel plate condenser is

$$C = \frac{Q}{V} = \frac{KS}{4\pi D} \qquad c.\,g.\,s. \text{ electrostatic units} \qquad (23)$$

where K is the dielectric constant of the dielectric between the plates, S the area of each plate and D the distance between the plates.

b. Spherical Condenser.—Another simple form of condenser consists of two concentric spherical shells, the space between which is filled with a uniform dielectric. Let the shells have the radii r_1 and r_2 and let the dielectric constant of the dielectric between the two shells be K. A charge of Q $c.\,g.\,s.$ electrostatic units given the inside sphere will induce a charge of $-Q$ units on the inside surface

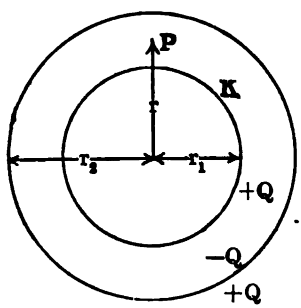


Fig. 78.

of the outer shell and a charge of $+Q$ units on the outside surface of this shell and whatever other conductors may be connected to it. This outside charge $+Q$, however, will have no effect on the field intensity inside this shell (see Article 137).

From symmetry, the lines of electrostatic induction are radial lines normal to the surfaces of two spheres; the total number coming out from the charge Q is $4\pi Q$ (see Article 141). Hence the electrostatic flux density at any point P in the dielectric is $B = \frac{4\pi Q}{4\pi r^2} = \frac{Q}{r^2}$ where r is the distance of P from the center of the spheres. Hence the field intensity at P is $H = \frac{B}{K} = \frac{Q}{Kr^2}$ and therefore the difference of potential between the two spheres is

$$V = \int_{r_1}^{r_2} \frac{Q}{Kr_2} dr = \frac{Q}{K}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Hence the capacity of this condenser is

$$C = \frac{Q}{V} = \frac{r_1 r_2}{r_2 - r_1} K \qquad c.\,g.\,s.\ \text{electrostatic units} \qquad (24)$$

c. **Coaxial Cylinders.** — Applying exactly similar reasoning to the case of two coaxial cylinders which are sufficiently long in comparison with their diameters so that the lines of electrostatic induction going out the ends may be neglected, we have from symmetry that the electrostatic flux density at any point P in the dielectric between the two cylinders is $B = \frac{4\pi Q}{2\pi r} = \frac{2Q}{r}$ where Q is the *charge per unit length* of the condenser, and r the distance of P from the center of the cylinders. Hence the field intensity at P is $H = \frac{2Q}{Kr}$ and therefore the difference of potential between the two cylinders is

$$V = \int_{r_1}^{r} \frac{2Q}{Kr} dr = \frac{2Q}{K} \ln \frac{r_2}{r_1}$$

where r_1 is the outside radius of the inside cylinder and r_2 the inside radius of the outside cylinder. Hence the capacity per centimeter of the condenser formed by two coaxial cylinders is

$$C = \frac{Q}{V} = \frac{K}{2 \ln \frac{r_2}{r_1}} \qquad c.\,g.\,s.\ \text{electrostatic units} \qquad (25)$$

This formula is applicable to a single wire enclosed in a lead sheath. For practical calculations this formula may be written

$$C=\frac{0.007354\ K}{log\frac{r_2}{r_1}} \quad \text{microfarads per 1000 feet} \qquad (25a)$$

or

$$C=\frac{0.03883\ K}{log\frac{r_2}{r_1}} \quad \text{microfarads per mile} \qquad (25b)$$

d. Two Parallel Cylinders. — In case of two parallel cylinders which are not coaxial, Fig. 79, but a distance D apart, *i.e.*, two parallel wires, an approximate formula for the capacity per unit

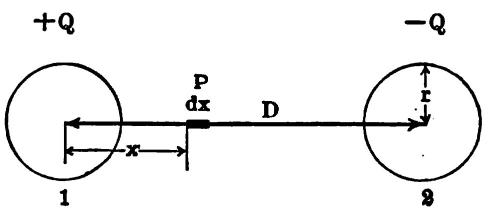


Fig. 79.

length, in case the cylinders are long compared to their distance apart, can be derived by assuming the charge on each cylinder is uniformly distributed over the surface of the cylinder. In this case the electrostatic intensity at any point P on the line joining the centers of the two cylinders due to the charge on 1 is $H_1=\frac{2\ Q}{Kx}$ and the electrostatic intensity at P in the same direction due to the charge $-Q$ on 2 is $H_2=\frac{2\ Q}{K\,(D-x)}$; therefore the total intensity at P is $\frac{2\ Q}{K}\left[\frac{1}{x}+\frac{1}{D-x}\right]$. Hence the difference of potential between 1 and 2 is

$$V=\frac{2\ Q}{K}\int_{r}^{D-r}\left[\frac{1}{x}+\frac{1}{D-x}\right]dx=\frac{4\ Q}{K}ln\frac{D-r}{r}$$

Hence the capacity per centimeter of the condenser formed by the two parallel wires when D is large in comparison with r is (approximately)

$$C = \frac{Q}{V} = \frac{K}{4\, ln \frac{D}{r}} \qquad c.\, g.\, s. \text{ electrostatic units} \qquad (26)$$

For practical calculations this formula may be written

$$C = \frac{0.003677\, K}{log \frac{D}{r}} \qquad \text{microfarads per 1000 feet} \quad (26a)$$

or

$$C = \frac{0.01941}{log \frac{D}{r}} \qquad \text{microfarads per mile} \quad (26b)$$

[Neither $\frac{D}{r}$ or $\frac{D-r}{r}$ is correct when the wires are close together, since the charge on the two wires is no longer uniformly distributed (see Alex. Russell, *Alternating Currents*, Vol. I, p. 99). When the wires are far apart r is negligible in comparison with D, hence formula (26) is sufficiently accurate.]

153. Specific Inductive Capacity. — In all the formulas for capacity it is seen that the capacity varies directly as the dielectric constant of the medium between the two conductors. Hence by measuring the capacity C_a of a given condenser when the plates are separated by air and then measuring the capacity C_k when the plates are separated by any other dielectric, the dielectric constant may be readily determined experimentally, since $K = \frac{C_k}{C_a}$. Since the numerical value of the dielectric constant is equal to the ratio of two capacities, this constant is frequently called the "specific inductive capacity" of the dielectric.

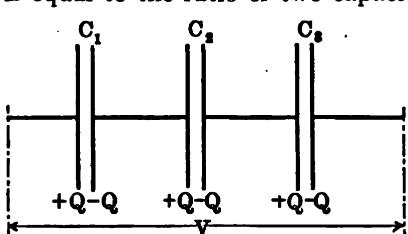


Fig. 80.

154. Condensers in Series and in Parallel. — When two or more condensers are connected in series, equal and opposite charges will be induced on each pair of plates connected by a conductor. Hence if the condensers have capacities C_1, C_2, C_3, etc., the total potential drop through all the condensers is

$$V=\frac{Q}{C_1}+\frac{Q}{C_2}+\frac{Q}{C_3}=Q\left(\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}\right)$$

Hence the equivalent capacity of any number of condensers in series is C_s where

$$\frac{1}{C_s}=\frac{V}{Q}=\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}+- - - \tag{27}$$

When the condensers are connected in parallel, the drop of potential across each condenser is the same, and therefore the

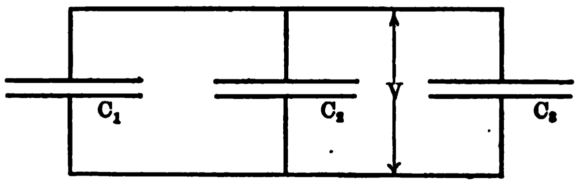


Fig. 81.

total positive charge given all the condensers (equal to the total negative charge) is

$$Q=(C_1+C_2+C_3)\ V$$

Hence the equivalent capacity of any number of condensers in parallel is

$$C_p=\frac{Q}{V}=C_1+C_2+C_3 \tag{28}$$

Compare with resistances in series and parallel, Article 98.

155. Absorption in a Condenser. — Residual Charge. — Experiment shows that when a given difference of potential is established between the plates of a condenser the dielectric of which is a solid, the charge taken by the condenser depends upon the length of time this potential difference is maintained. Again, when a charged condenser is discharged by connecting its plates momentarily with a conductor, and this connection is then broken, the plates at first appear to be entirely discharged, but after a few seconds a charge again appears on them resulting in the re-establishment of a difference of potential (less than in the first case) between the plates. In short, a solid dielectric apparently absorbs a certain amount of charge which it gives up only after a considerable lapse of time. The charge which appears on the plates of the condenser after the first discharge is called the "residual" charge; this phenomenon is called *electric absorption*. The absorption of a dielectric is apparently due to impurities in

it or to lack of homogeneity. It is greatest in substance like mica and glass; it is doubtful if absorption occurs at all in absolutely homogeneous substances.

156. Energy of an Electrostatic Field. — An electrostatic field represents a certain amount of stored energy just as a magnetic field represents a certain amount of stored energy. Let a condenser be charged by being connected to a battery or other source of electromotive force; the amount of energy stored in the electrostatic field can be readily calculated. Let the capacity of the condenser be C and let the drop of electrostatic potential from its positive to its negative plate at any instant be v. Then the charge on each plate of the condenser at this instant is numerically equal to Cv. To change the charge by an amount $dq = Cdv$ in time dt requires a current of $\frac{dq}{dt} = C\frac{dv}{dt}$ units for dt seconds. In the battery and the wires connecting the battery to the plates of the condenser this current will be in the direction from the negative to the positive plate of the condenser and in the dielectric separating the plates of the condenser this current (as a displacement current) will be in the direction from positive to the negative plate of the condenser, that is in the direction of the *drop* of potential through the condenser. Hence an amount of energy equal to $C\frac{dv}{dt}. v. dt = Cvdv$ will be lost by the current in the condenser. Hence, when the drop of potential between the two plates of the condenser changes by an amount dv an amount of energy equal to $Cvdv$ is gained by the electrostatic field in the dielectric separating the two plates. Therefore, when the difference of potential between the two plates is increased from zero to v a total amount of energy

$$W = \int_{o}^{v} Cvdv = \frac{1}{2} C\, v^2 = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\, q\, v \qquad (29)$$

is stored in the electrostatic field. In using these formulas, care must be taken to express all the quantities in the same system of units.

The energy which is stored in the electrostatic field comes from the source of electromotive force to which the condenser is connected. When the plates are disconnected from the source of electromotive force and are connected to each other by a wire,

the condenser discharges and the energy which was stored in the electrostatic field is transferred to the wire as heat energy. A charged condenser is then the seat of an electromotive force, and the direction of this electromotive force *through the condenser* is always in the direction from the negative plate of the condenser to the positive plate. Hence, when the condenser is being charged its electromotive force opposes the current, that is, the electromotive force of the condenser is a back electromotive force; when the condenser discharges, the electromotive force is in the direction of the current.

It can be shown by a similar process of reasoning to that employed in Article 125 that the total energy of any system of charged conductors may be expressed as the volume integral

$$W = \frac{1}{8\pi}\int_V K H^2 \, dv = \frac{1}{8\pi}\int_V \frac{B^2}{K} \, dv \qquad (30a)$$

throughout all space, where dv is any elementary volume, H the electrostatic intensity at dv, B the electrostatic induction at dv and K the dielectric constant of the dielectric in the space occupied by dv. Or, more briefly, the energy *per unit volume* of the electrostatic field is

$$w = \frac{K H^2}{8\pi} = \frac{B^2}{8\pi K} \qquad (30b)$$

157. Dielectric Hysteresis. — No substance is a perfect dielectric, but even the best insulators, such as porcelain or glass, are conductors to a slight extent. Hence when a constant electromotive force is impressed across a condenser, a small continuous current is established through the dielectric and therefore a certain amount of heat energy is dissipated in the dielectric. When the potential drop across the condenser is V and the conductance of the condenser is g, the rate at which heat energy is developed is (Joule's Law) equal to gV^2. When an alternating potential difference (see Chapter VI) of the same effective value is established across the condenser it is found that, when the dielectric of the condenser possesses the property of electric absorption, the heat energy dissipated in the condenser is greater than gV^2. This increase in the heat energy dissipated in the dielectric is sometimes said to be due to *dielectric hysteresis*. The phenomenon, however, is probably not due to a lag of the flux density behind the resultant field intensity, as in the case of magnetic hysteresis,

but is rather due to the existence in the dielectric of small conducting particles, insulated from one another, which form minute condensers and in which the varying electrostatic field sets up alternating currents of greater magnitude than would be set up by a constant impressed electromotive force. The increase of heat energy dissipated when an alternating potential difference is established across the condenser over that dissipated when a constant potential difference is established, is then due to the excess of heat energy developed by the alternating currents in these conducting particles.

SUMMARY OF IMPORTANT DEFINITIONS AND PRINCIPLES

(Note: All formulas are in *c. g. s.* electrostatic units unless otherwise specified.)

1. An **electrostatic unit point-charge** is a charge which repels with a force of one dyne an equal point-charge one centimeter away.

2. Two point-charges of q and q' units at a distance r centimeters apart repel each other with a force of

$$f=\frac{qq'}{r^2}$$

3. The **electrostatic field intensity** H at any point in an electrostatic field is the force in dynes which would act on a unit positive point-charge place at that point due solely to the agents producing the original field.

4. The field intensity at any point due to a point-charge q at a distance r centimeters away is

$$H=\frac{q}{r^2}$$

5. The mechanical force exerted on a point-charge q is

$$F=qH$$

where H is the field intensity due to every agent in the field except the charge q.

6. **Lines of electrostatic force** are lines drawn in an electrostatic field in such a manner that they coincide in direction at each point P with the field intensity at P and their number per unit area at each point P across a surface at right angles to their direction is equal to the field intensity at P.

7. **Gauss's Theorem.** — The algebraic sum of the number of lines of electrostatic force outward from any closed surface is equal to 4π times the algebraic sum of the charges inside this surface.

8. A **dielectric** in an electrostatic field is considered to be **made up of filaments** such that were the lateral walls of any one of these filaments separated from the rest of the dielectric by a narrow air gap, no charges would be induced on these walls.

9. The **intensity of electrisation** J at any point in a body is defined as the charge per unit area which would appear on a gap cut in the body at this point perpendicular to the line of electrisation through this point. The direction of the intensity of electrisation is the direction of the line of electrisation, the positive sense of which is the sense of the line drawn *into* the gap *from* the wall on which the positive charge is formed.

10. **A line of electrisation**, or unit filament, is a filament of such a size that were it broken by a narrow gap at any point, the numerical value of the charge formed on either wall of the gap would be $\frac{1}{4\pi}$.

11. The number of lines of **electrostatic induction**, or flux of electrostatic induction, crossing any surface is defined as the algebraic sum of the number of lines of electrisation and the lines of electrostatic force crossing that surface.

12. The **electrostatic flux density** B at any point is the number of lines of electrostatic induction per square centimeter crossing an elementary surface drawn at this point normal to their direction. The electrostatic flux density is the vector sum

$$B = \overline{4\pi J + H}$$

13. In an electrostatic field due solely to electric charges, a line of electrostatic force originates at a positive charge and ends at a negative charge, and may exist in any dielectric but not in a conductor in which no current is flowing. A line of electrisation originates at a negative *induced* charge on a dielectric and runs through the dielectric to a positive *induced* charge on the surface of the dielectric. A line of electrostatic induction is a continuous line which originates at a positively charged conductor (or at a positive *contact* charge on a dielectric) and ends at a negatively charged conductor (or at a negative *contact* charge on a dielectric), but passes through the *induced* charges on dielectrics.

14. The **dielectric constant**, or **specific inductive capacity**, K, of

a dielectric is the ratio of the electrostatic flux density B in the dielectric to the resultant electrostatic intensity H when the dielectric is placed in an electrostatic field; that is

$$K=\frac{B}{H}$$

15. A closed **hollow conductor** completely **screens** everything inside from external electrostatic influences.

16. The **electrostatic intensity within** the substance of a **conductor** in which there is no electric current is zero.

17. The total charge within any region completely enclosed by a conductor is zero.

18. The **electrostatic intensity** at any point P **just outside a charged conductor** in which there is no electric current is normal to the surface and equal to

$$H=\frac{4\pi\sigma}{K}$$

where K is the dielectric constant of the dielectric in contact with the conductor at P and σ is the surface density of the charge on the conductor just opposite P.

19. The **electrostatic flux density** at any point P **just outside a charged conductor** in which there is no electric current is normal to the surface and equal to

$$B=4\pi\sigma$$

independent of the dielectric in contact with the conductor; σ is the surface density of the charge on the conductor just opposite P.

20. The **electrostatic potential** V at any point in an electrostatic field is the work which would be done by the agents producing the field in moving a unit positive point-charge from this point to infinity. The electrostatic potential at any point due to a point-charge q at a distance r centimeters away is

$$V=\frac{q}{r}$$

The resultant electrostatic potential due to any number of charges is the algebraic sum of the potential due to all the individual poles.

21. The **drop of electrostatic potential** along any path from any point 1 to any point 2 is

$$V_1-V_2=\int_1^2 (H\cos\theta)\,dl$$

where dl is the length of any element of the path from 1 to 2, H the field intensity at dl and θ the angle between the direction of

H and dl. When the field is due solely to electrostatic charges the drop of potential is independent of the path from 1 to 2. The drop of electrostatic potential around a closed path is equal to the algebraic sum of the contact and induced electromotive forces in this path, provided all quantities are expressed in the same system of units.

22. An **electrostatic equipotential surface** is a surface drawn in an electrostatic field in such a manner that the drop of electrostatic potential along any path in the surface is zero. Such a surface is perpendicular at each point to the line of electrostatic force through that point.

23. When as the unit of electrostatic charge is taken a unit equal to 3×10^{10} times the size of an electrostatic unit, called the **electromagnetic unit of charge**, the charge produced at a conducting surface by a variable current is equal to the quantity of electricity (Article 80) transferred to that surface by the electric current in the conductor.

24. When as the unit of electrostatic potential difference is taken a unit equal to $\frac{1}{3\times10^{10}}$ times the size of the electrostatic unit of potential difference, called the **electromagnetic unit of electrostatic potential difference**, the electrostatic difference of potential between any two points is equal to the difference of electric potential (Article 87) in abvolts between these points.

25. When as the unit of electrostatic intensity is taken a unit equal to $\frac{1}{3\times10^{10}}$ times the size of the electrostatic unit of intensity, called the **electromagnetic unit of electrostatic intensity**, the electrostatic intensity at any point is equal to the electric intensity (Article 102) in abvolts per centimeter at this point.

26. The **displacement current** i_d flowing away from a conducting surface through the dielectric in contact with this surface is equal to the time rate of change of the charge q on this surface, *i.e.*,

$$i_d = \frac{dq}{dt}$$

where all quantities are in the same system of units. The current density of the displacement current at any point in a dielectric is

$$\frac{1}{4\pi}\frac{dB}{dt}$$

where B is the electrostatic flux density and all quantities are in the same system of units.

27. The value of the ratio of the displacement current between two equipotential surfaces in a dielectric to the time rate of change of the difference of potential between these two surfaces is called the **capacity** C of the portion of the dielectric between the two surfaces. The displacement current through a given portion of a dielectric between two equipotential surfaces is then

$$i_d = C\frac{dv}{dt}\cdot$$

where v is the difference of potential between the two surfaces. The *c. g. s.* electrostatic unit of capacity, the electromagnetic unit or abfarad and the practical unit or farad are related as follows:

1 abfarad $=10^9$ farads $=9\times 10^{20}$ *c. g. s.* electrostatic units.

When the two equipotential surfaces are conducting surfaces, the dielectric and the two conducting surfaces are said to form **an electric condenser.** The capacity of a condenser is also equal to the numerical value of the ratio of the charge on either surface to the difference of potential between the two.

28. The capacity of the condenser formed by **two long parallel wires** of circular cross section is

$$C = \frac{0.01941\ K}{log\ \frac{D}{r}} \qquad \text{microfarads per mile}$$

where K is the dielectric constant of the surrounding medium, D the distance between centers of wires and r the radius of each wire.

29. The resultant capacity of two or more **condensers in series** is the reciprocal of the sum of the reciprocals of the various capacities, *i.e.*,

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + - - - -$$

The resultant capacity of two or more **condensers in parallel** is the sum of the various capacities, *i.e.*,

$$C = C_1 + C_2 + - - -$$

30. The **energy stored in the dielectric** of a condenser is

$$W = \frac{1}{2}Cv^2 = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}qv$$

where C is the capacity of the condenser, q the numerical value of the charge on either conductor, and v the difference of potential

between the conductors. These formulas hold for any system of units provided all quantities are expressed in the same system.

PROBLEMS

1. A circular disc 20 inches in diameter and of negligible thickness is charged uniformly on both sides with a total charge of 15×10^5 electrostatic units. (1) What is the intensity of the electrostatic field at a point on a normal to the disc at its center 10 inches from the plane of the disc? (2) What would be the intensity at this point if the disc had an infinite radius and the surface density of the charge remains the same? (3) Under the latter conditions what would be the difference in electrostatic potential between the point and the disc?

Ans.: (1) 1358 *c. g. s.* electrostatic units; (2) 4650 *c. g. s.* electrostatic units; (3) 118,000 *c. g. s.* electrostatic units.

2. Two plates A and B, the areas of which may be considered to be infinite, are placed parallel to each other and 10 cm. apart. A is charged positively with 50 electrostatic units of charge per sq. cm. while B is charged negatively with the same density; the charges are confined to the surfaces which face each other. (1) Determine the pull per sq. cm. which A exerts upon B.

A slab of glass of infinite area and 8 cm. in thickness is now placed between the two plates and parallel to them, but not touching either. The dielectric constant of the glass is 5. (2) Determine again the pull per sq. cm. which A exerts upon B; (3) the intensity of electrisation in the glass; and (4) the electrostatic flux density in the glass.

Ans.: (1) 15,700 dynes; (2) 15,700 dynes; (3) 40 *c. g. s.* electrostatic units; (4) 629 *c. g. s.* electrostatic units.

3. Determine the difference of electrostatic potential between the two plates in Problem 2 when the plates are charged positively and negatively respectively, with 0.05 *c. g. s.* electrostatic units per sq. cm. (1) when the medium between the plates is air; (2) when a slab of glass of infinite area, 6 cm. in thickness and with a dielectric constant equal to 5, is placed between the plates and parallel to them. The charge on the plates and the distance between the plates remain constant in the two cases.

If the difference of potential between the two plates is kept constant at 0.1 electrostatic unit, what will be the numerical value of the charge per sq. cm. on each plate (3) when the medium be-

tween the plates is air; (4) when the slab of glass 6 cm. thick is placed between the plates?

Ans.: (1) 6.29 *c. g. s.* electrostatic units; (2) 3.27 *c. g. s.* electrostatic units; (3) 0.000795 *c. g. s.* electrostatic units; (4) 0.00153 *c. g. s.* electrostatic units.

4. Two parallel metallic plates each 100 sq. cm. in area are separated by a distance of 0.5 cm. If the difference of potential between the two plates is maintained at 1000 volts, determine (1) the charge in microcoulombs on each plate when the plates are separated by air; (2) when the dielectric between the two plates is glass; (3) the work required to pull the glass out of the electrostatic field. The dielectric constant of glass is 5. Assume a uniform distribution of charge.

Ans.: (1) 0.01768 microcoulombs; (2) 0.0884 microcoulombs; (3) 354 ergs.

5. Two metallic plates each 100 sq. cm. in area are charged until the difference of potential between the plates is 300 volts and the source of potential is then removed. The plates are 0.5 cm. apart and the medium between them is air. (1) Determine the work done in separating the plates until they are one centimeter apart. (2) What is the difference of potential between the plates if a sheet of glass 0.5 cm. thick and having a dielectric constant of 5 is then pushed in between them?

Ans.: (1) 7.96 ergs; (2) 360 volts.

6. A transmission line 10 miles in length consists of two No. 0000 wires (B. & S. gauge) spaced 3 feet between centers. If a potential difference of 1000 volts is established between the wires and the line is open at the far end, determine the energy in the electrostatic field surrounding the line. The diameter of a No. 0000 wire is 0.460 inch.

Ans.: 0.0443 joules.

7. The total inductance of a two-wire transmission line 25 miles in length is 84.6 millihenries. What is its capacity in microfarads?

Ans.: 0.223 microfarads.

8. Three condensers, A, B and C, the capacities of which are 25, 20 and 15 microfarads respectively, are connected in series. If the potential drop across B is 100 volts, determine (1) the potential drop across A; (2) the potential drop across C; and (3) the potential drop across the three condensers in series. (4) What is

the charge in microcoulombs on each condenser; and (5) the capacity of the three condensers in series?

Ans.: (1) 80 volts; (2) 133.3 volts; (3) 313.3 volts; (4) 2000 microcoulombs; (5) 6.38 microfarads.

9. When an *e. m. f.* of 200 volts is impressed across two condensers A and B of unknown capacity connected in series, the potential drop through A is three times that through B and the electrostatic energy of the system is 0.5 joule. What would be the charge upon A and upon B when an *e. m. f.* of 100 volts is impressed across the condensers in parallel?

Ans.: 0.00333 and 0.01 coulomb respectively.

10. A lead-covered cable is made of a No. 00 B. & S. wire surrounded by a layer of rubber 0.25 inch thick, which is in turn surrounded by a layer of gutta-percha 0.25 inch thick, the whole being encased in the lead sheath. The specific inductive capacity of the rubber and the gutta-percha is 2.2 and 4.5 respectively. What is the ratio of the potential drop through the rubber to that through the gutta-percha when a given difference of potential is established between the wire and the sheath? The resistance of the dielectric is to be considered infinite. The diameter of a No. 00 wire is 0.365 inch.

Ans.: 3.87.

11. What is the displacement current per mile of the above cable when the difference of potential between wire and sheath varies at the rate of 10,000 volts per second?

Ans.: 1810 microamperes.

VI

VARIABLE CURRENTS

158. Total Energy Associated with an Electric Circuit. — From the foregoing discussion of the magnetic and electrostatic field it is evident that the energy associated with an electric circuit may manifest itself in the following ways:

1. As magnetic energy, due to the magnetic field produced by the current in the circuit.

2. As electrostatic energy, due to the electrostatic field produced by the differences of electric potential between the various parts of the circuit.

3. As mechanical energy, due to the relative motion of the various parts of the circuit or to the motion of conductors or of magnetic or dielectric bodies in the surrounding field.

4. As chemical energy, due to chemical changes which may take place in various parts of the circuit.

5. As reversible heat energy, due to thermal electromotive forces.

6. As heat energy dissipated in the conductors (and to a slight extent in the surrounding dielectric), due to their electric resistance.

7. As heat energy dissipated in surrounding magnetic bodies due to the variation of the magnetic flux through them; *i.e.*, magnetic hysteresis.

8. As heat energy dissipated in the surrounding dielectric bodies due to the variation of the electrostatic flux in them; *i.e.*, dielectric hysteresis.

The effects produced by the first five forms of energy are all reversible; *i.e.*, energy is required to establish a magnetic or an electrostatic field, but the same amount of energy is given back in some other form when these fields disappear; energy is required to move a conductor or a magnetic or a dielectric body in the field, but the same amount of energy is given back in some other form when the motion of the body is reversed; energy is required to produce any chemical action but the same amount of energy is

given back in some other form when this action is reversed; the heat energy given out at a junction of two dissimilar substances when the current flows through the junction in one direction, is given back as some other form of energy when the current is reversed.

The effects produced by the last three forms of energy are *not* reversible; *i.e.*, energy is always dissipated in a conductor through which a current is flowing, independent of the direction of the current; energy is always dissipated when the phenomenon of magnetic or dielectric hysteresis occurs, independent of whether the flux is increasing or decreasing.

The first five effects are sometimes said to be due to *conservative* forces, the last three to *dissipative* forces. The word forces is here used in a general sense, meaning the something that produces or opposes the effects.

159. General Equations of the Simple Electric Circuit. — The transfer of energy from one region to another in space by means of an electric current is in general accompanied by production of energy in all these various forms, except in the special case of a continuous current, *i.e.*, a current which does not vary with time; there is also in general a transfer of energy from one form to another all along the circuit. To understand clearly the effects produced by a variable current it is necessary to confine one's attention at first to comparatively simple circuits.

Consider a circuit consisting of a coil having a resistance r and an inductance L in series with a condenser having a capacity C and a conductance g. The conductance of a condenser, that

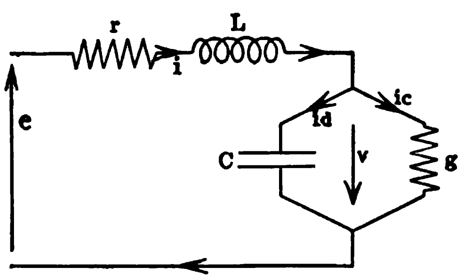


Fig. 82.

is, the reciprocal of the resistance of the dielectric between its plates, is sometimes called the *leakance* of the condenser. This circuit is typical of the circuits one has to deal with in practice;

in fact the most general form of circuit may be looked upon as made up of such elementary circuits. This simple circuit may be represented diagrammatically as shown in Fig. 82, the resistance and inductance and the leakance and capacity being shown separately simply for convenience. Let an electromotive force e be impressed across the terminals of this circuit. This impressed electromotive force will establish a current in the circuit and therefore a potential drop across the various parts of the circuit. When there is originally no current in the circuit and no charge on the condenser, this current will be around the circuit in the direction of the impressed electromotive force, and will produce a drop of potential through the coil and through the condenser. The current in the coil will have the same value at each part of its winding at any given instant (provided the capacity and leakance of the coil may be neglected) and this current must in turn be equal to the total current through the condenser. Let i be the current in the coil, i_d the displacement current through the condenser and i_c the conduction current through the condenser. Then from Article 149

$$i = i_d + i_c$$

but $i_d = \frac{dq}{dt}$, where $\frac{dq}{dt}$ represents the time rate of change of the charge on the condenser. Let v be the potential drop through the condenser in the direction of the current through it, then from Article 149

$$i_d = C\frac{dv}{dt}$$

The conduction current through the condenser is, by Ohm's Law, $i_c = gv$. Hence

$$i = gv + C\frac{dv}{dt} \tag{1}$$

From Ohm's Law, the drop of potential through the coil due to its resistance is ri. From Article 116 the drop of potential through the coil due to its inductance (*i.e.*, the back electromotive force of induction) is $L\frac{di}{dt}$. Hence the total drop of potential through the coil is

$$ri + L\frac{di}{dt}$$

and therefore the total drop of potential through the coil and the condenser in series is

$$ri + L\frac{di}{dt} + v$$

and this must be equal to the impressed electromotive force e, that is,

$$e = ri + L\frac{di}{dt} + v \tag{2}$$

Equations (1) and (2) are two equations in the two unknown quantities i and v. The solution of practically every electrical engineering problem that has to do with the transmission of energy by variable or alternating currents is simply a special solution of one or more sets of equations of this kind. It is therefore necessary to understand thoroughly the physical meaning of every term in these two equations and to become thoroughly familiar with the various mathematical expedients employed in their solution.

Since energy is transferred by the current to the electrostatic field of the condenser, the potential drop v across the condenser may be looked upon as the back electromotive force of the condenser (see Article 156).

These two equations may then be looked upon simply as a special case of Kirchhoff's two laws. In the first equation i represents the current flowing to the junction between the coil and condenser and $gv + C\frac{dv}{dt}$ represents the total current flowing away from this junction. In the second equation, which may be written $e - v - L\frac{di}{dt} = ir$, the symbols e, v, and $L\frac{di}{dt}$ each represent an *e. m. f.*, the last two being in the *opposite* direction to the impressed electromotive force e, and therefore $e - v - L\frac{di}{dt}$ represents the sum of the *e.m.f.*'s in the loop formed by the source of the impressed *e. m. f.*, the inductance L, the resistance r, and the capacity C, and ir is the total resistance drop in this loop. In fact, Kirchhoff's Laws apply not only to steady currents but to currents varying in any manner whatever, *provided the instantaneous values of the currents are considered.*

It should be clearly borne in mind that equations (1) and (2)

apply directly only to a *single* circuit and when there is *no dissipation of energy due to either magnetic or dielectric hysteresis* or to currents induced in surrounding bodies. When there is any dissipation of energy due to hysteresis or eddy currents* a term should be added to take into account this dissipation of energy. In case there is iron or any other magnetic substance in the magnetic field produced by the current there are both magnetic hysteresis and eddy currents; in such a case equations (1) and (2) give only a first approximation to the true relations between current, potential difference and time; this approximation, however, is usually sufficiently close for practical work. We shall see in Article 190 how a still closer approximation can be made in the case of alternating currents.

In this chapter will be given the solutions of equations (1) and (2) for a few simple cases. In every case the inductance will be assumed constant; when there is iron in the magnetic field of the current this assumption will give only a roughly approximate solution.

160. Establishment of a Steady Current in a Circuit Containing Resistance and Inductance. — The circuit is represented diagrammatically in Fig. 83; E represents a *constant e. m. f.* Let the time t be reckoned from the instant the switch S is closed; *i.e.*, let the switch be closed at time $t=0$. At any instant an interval of time t after closing the circuit.

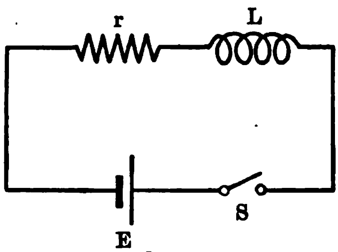


Fig. 83.

$$E=ri+L\frac{di}{dt}$$

whence

$$\frac{r}{L}dt=-\frac{d\,(E-ri)}{E-ri}$$

Integrating,

$$\frac{rt}{L}=-\ln\,(E-ri)+G$$

where G is a constant of integration. The value of G is found by

* The name *eddy current* is applied to the currents induced in the magnetic circuit or in any metal forming the frame of an electric machine.

the condition that at time $t=0$ (that is, at the instant the circuit is closed) $i=0$.* Substituting these values in the above equation gives

$$G = ln\ E$$

hence at any instant t seconds after closing the circuit

$$-\frac{rt}{L} = ln\left(\frac{E-ri}{E}\right)$$

Writing each side of this equation as the exponent of the base ϵ of the natural system of logarithms, we get

$$\epsilon^{-\frac{rt}{L}} = \frac{E-ri}{E}$$

whence

$$i = \frac{E}{r}\left(1-\epsilon^{-\frac{rt}{L}}\right) \tag{3}$$

The physical interpretation of this equation is that the current reaches its steady value $I=\frac{E}{r}$ only after the time t measured from the instant of closing the circuit has become sufficiently great to make the term $\epsilon^{-\frac{rt}{L}}$ sensibly equal to zero. Since, however, the ratio $\frac{r}{L}$ is usually quite large, this term $\epsilon^{-\frac{rt}{L}}$ as a rule becomes practically zero for t equal to a small fraction of a second, and therefore the current reaches its steady value $I=\frac{E}{r}$ almost immediately after the circuit is closed. When the self-induction is large, as in the case of a coil wound on a closed iron core, the ratio $\frac{r}{L}$ may be relatively small, in which case several seconds may elapse before the current reaches its steady value.

In any case, after an interval of time $T=\frac{L}{r}$, the current is

$$i_T = \frac{E}{r}\left(1-\epsilon^{-1}\right) = \frac{E}{r}\left(1-\frac{1}{2.718}\right) = 0.632\ \frac{E}{r}$$

That is, after an interval of time $T=\frac{L}{r}$, the current reaches 63.2

*Otherwise at the instant the switch is closed energy would be transferred to the magnetic field at an *infinite* rate, which is impossible.

per cent of its final value. The time required for the current in such a circuit to reach $100\,(1-\epsilon^{-1}) = 63.2$ per cent of its final steady value when a steady *e. m. f.* is impressed upon the circuit is called the "time-constant" of the circuit. The time-constant of such a circuit may be looked upon as a measure of the slowness with which the current reaches its steady value, the greater the time-constant the longer the interval before the steady value is reached. For a circuit consisting of an inductance L and a resistance r the time-constant is equal to $\frac{L}{r}$. It should be noted that this L and r refer to the *entire* circuit; hence when the impressed *e. m. f.* is produced by a generator developing a steady armature *e. m. f.* E, the resistance and inductance of the generator must also be included. In addition, the above formulas hold only in case the inductance of the entire circuit is a constant, and there is no energy dissipated in hysteresis, which conditions never hold

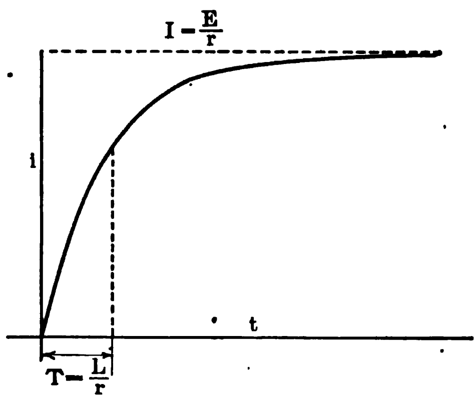


Fig. 84.

when there is iron or any other magnetic substance in the magnetic field produced by the current. In such cases, however, the formulas may be looked upon as a first approximation to the true relation between current and time.

The relation between current and time given by equation (3) may be represented graphically by plotting the current i as the ordinates against the time t as abscissas. The curve has the general shape shown in figure 84; *i.e.*, the current rises rapidly at first and becomes asymptotic to the line corresponding to

$I = \frac{E}{r}$, which gives the final steady value of the current. The abscissa T of the point having the ordinate $i = 0.632 \frac{E}{r}$ is equal to the time-constant of the circuit.

161. Decay of Current in a Circuit Containing Resistance and Inductance. — Let the circuit be short-circuited on itself at any instant when the current has the value i_0, say; let the time t be measured from this instant, *i.e.*, at time $t = 0$ let $i = i_0$. The circuit is represented diagrammatically in Fig. 85. In this case the impressed *e. m. f.* is zero, whence

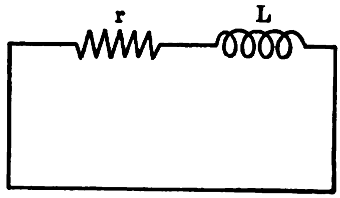


Fig. 85.

$$0 = ri + L \frac{di}{dt}$$

and therefore

$$-\frac{r\,dt}{L} = \frac{di}{i}$$

which integrated gives

$$-\frac{rt}{L} = \ln i + G$$

where G is a constant of integration. The value of G is found from the condition that at time $t = 0$, $i = i_0$ which values substituted in the above equation give $G = -\ln i$. Hence at any instant t seconds after closing the circuit

$$-\frac{rt}{L} = \ln \frac{i}{i_0}$$

whence

$$\epsilon^{-\frac{rt}{L}} = \frac{i}{i_0}$$

and therefore

$$i = i_0 \epsilon^{-\frac{rt}{L}} \qquad (4)$$

The physical interpretation of this equation is that the current does not fall to zero immediately, but only after a sufficient time t has elapsed to make the term $\epsilon^{-\frac{rt}{L}}$ sensibly zero, which is usually only a fraction of a second, unless the self-induction of the circuit

is large compared with its resistance. The current falls to 100 ϵ^{-1} = 36.8 per cent of its original value in the time $T = \frac{L}{r}$ = the time-constant of the circuit. The relation between the current i and

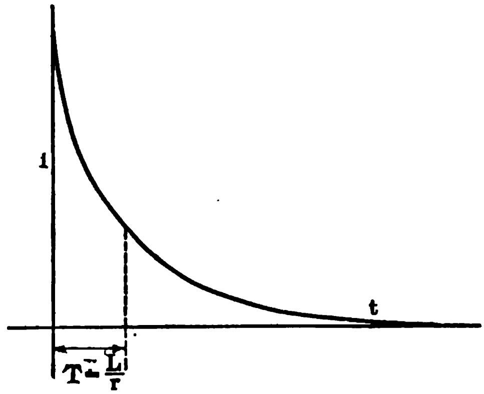


Fig. 86.

time t is shown graphically in Fig. 86. The current falls rapidly at first and becomes asymptotic to the axis of t, *i.e.*, to the line corresponding to $i = 0$.

162. Charging a Condenser through a Resistance.—The circuit is represented diagrammatically in Fig. 87; E is a *constant e. m. f.* Let the charge on the condenser be zero at the instant the switch is closed, and let time be measured from this instant; *i.e.*, at time $t = 0$ let $q_0 = 0$ and $i = 0$. Let q and i be the charge and current respectively and v the *p.d.* through the condenser at any instant t seconds after closing the switch. Then at this instant, *assuming the ideal conditions of no leakance in the condenser and no inductance of the circuit,*

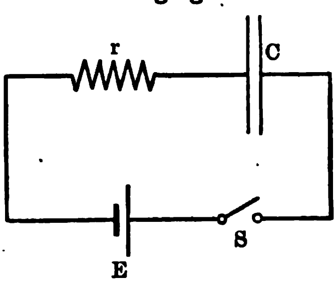


Fig. 87.

$$E = ri + v$$

$$i = C\frac{dv}{dt}.$$

From the first equation we have that $i = \frac{E - v}{r}$ which substituted in the second equation gives

$$\frac{dt}{rC}=\frac{dv}{E-v}=\frac{-d\,(E-v)}{E-v}$$

which integrated gives

$$-\frac{t}{rC}=ln\,(E-v)+G$$

where G is a constant of integration. For $t=0$, $v=0$,* which substituted in this equation gives $G=-\,ln\,E$. Substituting this value of G in the last equation gives the equation

$$-\frac{t}{rC}=ln\left(\frac{E-v}{E}\right)$$

whence

$$v=E\left(1-\epsilon^{-\frac{t}{rC}}\right) \qquad (5a)$$

which substituted in the second equation above gives

$$i=\frac{E}{r}\epsilon^{-\frac{t}{rC}} \qquad (5b)$$

and since $q=Cv$ at any instant we get immediately from (5*a*) that

$$q=C\,E\left(1-\epsilon^{-\frac{t}{rC}}\right). \qquad (5c)$$

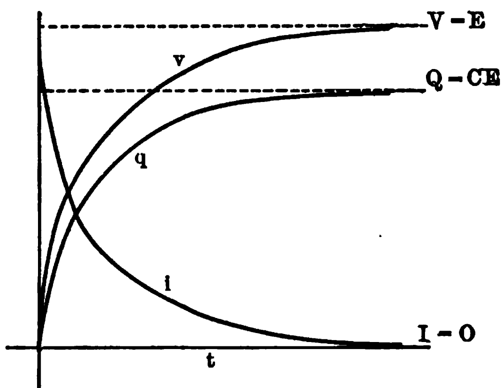


Fig 88.

Equations (5) then give the values of the *p.d.* through the condenser, the current, and the charge at any instant. The variation of v, i, and q with time is shown graphically in Fig. 88. Note that both the *p.d.* through the condenser and the charge start at

*Otherwise at the instant the switch is closed energy would be transferred to the electrostatic field at an *infinite rate*, which is impossible.

zero rise rapidly at first and then approach more slowly their constant values E and CE respectively. The current, on the other hand, has its maximum value $\frac{E}{r}$ at the instant the switch is closed and then falls, first rapidly and then more slowly, to zero.* As a rule rC is extremely small, so that only a small fraction of a second is required for these steady conditions to become established.

The product rC measures the slowness with which the condenser becomes charged; it is called the time-constant of the circuit. Compare with the time-constant of a circuit formed by a resistance and an inductance in series.

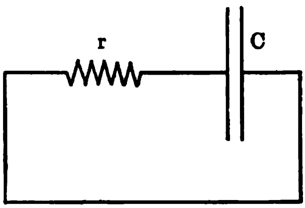


Fig. 89.

163. Discharging a Condenser through a Resistance. — Let the condenser be charged to a $p.d. v_0$ at the instant $t=0$ at which it is short-circuited through a resistance r, Fig. 89. *Assuming as before the ideal conditions of no leakance and no inductance*, we have

$$0=ri+v$$
$$i=C\frac{dv}{dt}$$

Eliminating i from these two equations gives

$$-rC\frac{dv}{dt}=v$$

whence

$$-\frac{dt}{rC}=\frac{dv}{v}$$

whence

$$-\frac{t}{rC}=ln\ v+G$$

But at $t=0$, $v=v_0$, whence $G=-lnv_0$.
Hence at any instant t seconds after the condenser is short-circuited,

*Actually the current always starts from zero and rises to a maximum value, due to the inductance of the circuit, which, though it may be small, is never absolutely zero as assumed in the above discussion; see Article 201.

$$v = v_0 \epsilon^{-\frac{t}{rC}} \qquad (6a)$$

$$i = -\frac{v_0 \epsilon^{-\frac{t}{rC}}}{r} \qquad (6b)$$

$$q = C v_0 \epsilon^{-\frac{t}{rC}} \qquad (6c)$$

In this case both the *p.d.* and the charge decrease at first rapidly and then more slowly to zero. The current through the condenser, however, is negative, *i.e.*, in the *opposite* direction to the drop of potential through the condenser. Hence the current through the resistance is in the direction from the + to the − plate of the condenser. The current is a maximum $\frac{v_0}{r}$ (in the negative direction) at the instant of short circuit,* and then decreases in the same manner as the *p.d.* and the charge.

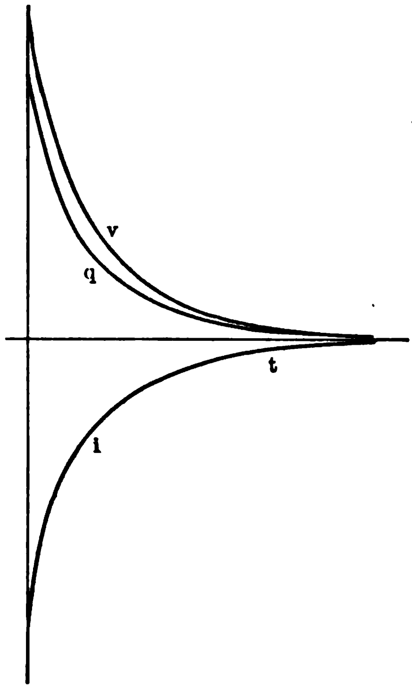


Fig. 90.

164. Discharge of a Condenser through an Inductance. — The circuit is represented diagrammatically in Fig. 91. The condenser is charged to a $p.d.v_0$ and at time $t=0$ the switch is closed. At any time t seconds later *assuming the ideal conditions of no leakance and no resistance,*

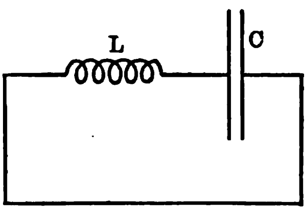


Fig. 91.

$$0 = L\frac{di}{dt} + v \qquad (a)$$

$$i = C\frac{dv}{dt} \qquad (b)$$

From equation (b) we have that $\frac{di}{dt} = C\frac{d^2v}{dt^2}$ which substituted in the first equation gives

* The current actually starts at zero due to the inductance of the circuit; see Article 201.

$$\frac{d^2v}{dt^2} = -\frac{1}{LC}v \qquad (c)$$

This equation is of exactly the same form as the equation for the pendulum; see Article 23. Compare also with the equation for the vibration of a magnet in a uniform magnetic field, Article 42. Its solution is

$$v = A \sin\left(\frac{t}{\sqrt{LC}} + \theta\right) \qquad (d)$$

which substituted in equation (*b*) gives

$$i = A\sqrt{\frac{C}{L}} \cos\left(\frac{t}{\sqrt{LC}} + \theta\right) \qquad (e)$$

The constants A and θ are determined from the condition that at time $t=0$, $v=v_0$, and $i=0$. Substituting these values in (*d*) and (*e*) we get

$$v_0 = A \sin \theta$$

$$0 = A\sqrt{\frac{C}{L}} \cos \theta$$

Whence $\theta = \frac{\pi}{2}$ and $A = v_0$. The equations for the *p.d.* and current at any instant are then

$$v = v_0 \cos\left(\frac{t}{\sqrt{LC}}\right) \qquad (7a)$$

$$i = -v_0\sqrt{\frac{C}{L}} \sin\left(\frac{t}{\sqrt{LC}}\right) \qquad (7b)$$

These two equations are plotted in Fig. 92. That is, both v and i are harmonic functions, or "sine waves," having a period equal to $2\pi\sqrt{LC}$. The maximum value of the *p.d.* is v_0 and the maximum value of the current is $v_0\sqrt{\frac{C}{L}}$. That is, the charge on the condenser oscillates from $q = Cv_0$ to $-q = -Cv_0$, and the current oscillates between the values $v_0\sqrt{\frac{C}{L}}$ and $-v_0\sqrt{\frac{C}{L}}$, the charge reaching the maximum when the current is zero and *vice versa*.

If the current in the inductance had been equal to i_0 and the *p.d.* across the condenser zero at time $t=0$, then the equations for *p.d.* and current would have been

$$v=i_0\sqrt{\frac{L}{C}}\sin\left(\frac{t}{\sqrt{LC}}\right) \tag{8a}$$

$$i=i_0\cos\left(\frac{t}{\sqrt{LC}}\right). \tag{8b}$$

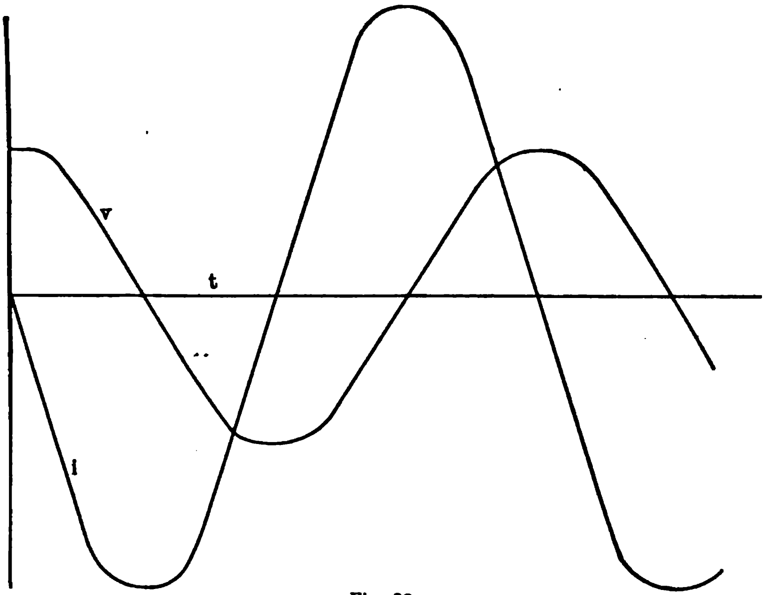


Fig. 92.

A circuit of this kind is approximately realized in the case of a transmission line, except that the inductance and capacity instead of being lumped in a single coil and a single condenser are

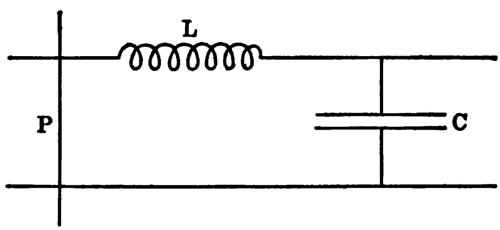


Fig. 93.

distributed uniformly along the line, and the resistance of the line is not negligible. As a first approximation we may consider the line as equivalent to a lumped inductance and a lumped capacity as shown in Fig. 93 and neglect the resistance. If the line, open

at the receiving end, becomes short-circuited at any time P, by a wire falling across it, for example, we have the conditions just discussed.

Take the case of a line consisting of two number 0000 B. & S. wires (solid) spaced 6 feet apart; let the line be 50 miles long (100 miles of wire).

The inductance of the line will then be 0.193 henries and the capacity 0.389×10^{-6} farads. Let 500 amperes be flowing in the line at the instant of short-circuit at P and let the *p.d.* across the end of the line (*i.e.*, across the condenser) be zero at this instant. Then this *p.d.* will reach a maximum of

$$500\sqrt{\frac{0.193}{0.389 \times 10^{-6}}} = 352{,}000 \text{ volts}$$

and the current and *p.d.* will begin to oscillate with a period of $2\pi\sqrt{LC} = 2\pi\sqrt{0.193 \times 0.389 \times 10^{-6}} = 0.00172$ seconds or 580 complete swings or cycles per second. This abnormal rise in voltage would of course puncture the insulators carrying the wires were not some sort of "safety valve" provided. Such a safety device is the lightning arrester, which in its simplest form consists of one or more spark gaps so arranged that when the voltage on the line rises a predetermined amount above normal, a spark jumps across the gap and thus reduces the voltage. Protecting a line against lightning discharges is only one of the functions of a lightning arrester; such a device is likewise necessary to protect the line (and the apparatus connected therewith) against the abnormal voltage which may be produced when heavy loads are switched on or off the line.

Note that the ideal case of a condenser without leakance and an inductance without resistance, which we have been considering, cannot be completely realized in practice, since the dielectric of every condenser is a conductor to a certain extent (though usually an extremely poor conductor) and every circuit made of conductors has a certain resistance. The effect of resistance and leakance is to damp out the oscillations of the electric current in the same way that friction damps out the oscillation of a vibrating pendulum. (See Article 202.)

SUMMARY OF IMPORTANT RELATIONS

1. In general, the **energy associated with an electric current** may manifest itself in eight different forms, *i.e.*, as

Magnetic energy,
Electrostatic energy,
Mechanical energy,
Chemical energy,
Heat energy at the junction of dissimilar substances,
Heat energy due to the resistance of the conductors,
Heat energy due to magnetic hysteresis,
Heat energy due to dielectric hysteresis.

2. The **general equations of a circuit formed by a coil and a condenser in series** are

$$i = gv + C\frac{dv}{dt}$$

$$e = ri + L\frac{di}{dt} + v$$

where e is the impressed *e. m. f.* across the terminals of the circuit, v the potential drop through the condenser, i the current in the coil, r the resistance of the coil, L the inductance of the coil, g the leakance of the condenser and C the capacity of the condenser. These two equations are simply Kirchhoff's Law applied to the instantaneous values of the current, *e. m. f.* and *p. d.*

3. The current i in a circuit formed by a resistance r and an inductance L, t seconds after impressing a constant *e. m. f.* E across its terminals, is

$$i = \frac{E}{r}\left(1 - \epsilon^{-\frac{rt}{L}}\right)$$

provided there is originally no current in the circuit; r is the resistance, L the inductance of the *entire* circuit.

4. The **time-constant** of a circuit formed by a resistance r and an inductance L in series is $\frac{L}{r}$.

5. The current i in a circuit formed by a resistance r and an inductance L, t seconds after the terminals of the circuit are short-circuited, is

$$i = i_o\,\epsilon^{-\frac{rt}{L}}$$

where i_o is the current in the circuit at the instant of short-circuit, provided there is no *e. m. f.* in the circuit.

6. When a condenser of capacity C is charged through a resistance r by a constant impressed *e. m. f.* E, the *p. d.* across the condenser, the current in the circuit and the charge on the condenser t seconds after the *e. m. f.* is impressed, are respectively

$$v = E\left(1 - \epsilon^{-\frac{t}{rC}}\right)$$

$$i = \frac{E}{r}\,\epsilon^{-\frac{t}{rC}}$$

$$q = CE\left(1 - \epsilon^{-\frac{t}{rC}}\right)$$

provided there is originally no charge on the condenser and no current in the circuit and there is no inductance in any part of the circuit.

7. The time-constant of a circuit formed by a resistance r and a capacity C in series is rC.

8. When a condenser of capacity C, through which the *p.d.* is v_o, is discharged through a resistance r, the *p.d.* through the condenser, the current in the circuit and the charge on the condenser t seconds after the terminals of the circuit are short-circuited are respectively

$$v = v_o\,\epsilon^{-\frac{t}{rC}}$$

$$i = -\frac{v_o}{r}\,\epsilon^{-\frac{t}{rC}}$$

$$q = C v_o\,\epsilon^{-\frac{t}{rC}}$$

provided there is no current in the circuit at the instant of short-circuit and there is no *e. m. f.* and no inductance in the circuit.

9. When a condenser of capacity C, through which the *p.d.* is v_o, is discharged through an inductance L, the *p.d.* through the condenser and the current in the circuit t seconds after the terminals of the circuit are short-circuited are respectively

$$v = v_o \cos\left(\frac{t}{\sqrt{LC}}\right)$$

$$i = -v_o\sqrt{\frac{C}{L}}\,\sin\left(\frac{t}{\sqrt{LC}}\right)$$

provided there is no current in the circuit at the instant of short-circuit and there is no *e. m. f.* and no resistance in the circuit. When the terminals of the circuit are short-circuited at the instant

when the current is i_0 and there is no *p.d.* through the condenser, the *p.d.* and current t seconds after the terminals are short-circuited are respectively

$$v = i_0\sqrt{\frac{L}{C}}\,sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$i = i_0\,cos\left(\frac{t}{\sqrt{LC}}\right)$$

provided there is no *e. m. f.* and no resistance in the circuit.

PROBLEMS

1. An air-core solenoid of 5000 turns is 20 cm. long and has a diameter of 5 cm.; the resistance of the solenoid is 2 ohms. When a constant *e. m. f.* of 50 volts is impressed across the terminals of this coil (1), at what rate does the current begin to increase; (2) what is the final value of the current; and (3) what is the time-constant of the coil?

Ans.: (1) 162.2 amperes per second; (2) 25 amperes; (3) 0.154 second.

2. In the preceding problem, (1) at what rate is energy being stored in the magnetic field of the current when the current has reached a value of 10 amperes; (2) when the energy of the electromagnetic field is 30.8 joules at what rate is energy dissipated in heating the coil; (3) at what rate is the current changing when the total power supplied to the coil is 250 watts?

Ans.: (1) 300 watts; (2) 400 watts; (3) 129.9 amperes per second.

3. At the same instant that the *e. m. f.* impressed upon a coil is removed, the ends of the coil are connected by a resistance of 3 ohms. At the instant immediately after this change in connections is made the current in the coil is 12 amperes and is decreasing at a rate of 480 amperes per second. (1) What is the time-constant of the circuit? (2) If the energy of the electromagnetic field is 7.2 joules at the instant that the change in connections is made, at what rate is energy dissipated in the circuit in the form of heat energy at that instant? (3) What is the resistance and the inductance of the coil?

Ans.: (1) 0.025 second; (2) 576 watts; (3) resistance 1 ohm, inductance 0.1 henry.

4. An *e. m. f.* of 20 volts is impressed upon a coil of 0.6 ohm

resistance and 0.3 henry inductance. What is the value of the current in the coil 0.5 second after the *e. m. f.* is impressed?

Ans.: 21.1 amperes.

5. An *e. m. f.* of 30 volts is impressed upon a coil of 0.5 ohm resistance and 0.2 henry inductance. One tenth of a second after the *e. m. f.* is impressed, the source of the *e. m. f.* is removed and the coil is short-circuited by a resistance of 0.1 ohm. Determine the value of the current in the circuit 0.08 second later.

Ans.: 10.48 amperes.

6. A battery which has an *e. m. f.* of 10 volts and a resistance of 0.5 ohm, a coil which has an inductance of 3 henries and a resistance of 0.5 ohm, and a non-inductive resistance of 4 ohms are all connected in series to form a closed electric circuit. Find the value of the current flowing in a non-inductive resistance of $\frac{4}{7}$ of an ohm 2 seconds after it is connected in parallel with the 4 ohm resistance.

Ans.: 4.34 amperes.

7. An *e. m. f.* of 250 volts is impressed upon a circuit conformed by a non-inductive resistance of 1000 ohms in series with a condenser of 50 microfarads capacity. Find (1) the initial rate at which the condenser is charged; (2) the current in the circuit when the charge on the condenser is 5000 microcoulombs; (3) the energy in the electrostatic field when energy is dissipated as heat energy in the circuit at the rate of 10 watts; and (4) the charge in microcoulombs on the condenser when the current is changing at the rate of 4 amperes per second.

Ans.: (1) 0.25 ampere; (2) 0.15 ampere; (3) 0.563 joule; (4) 2500 microcoulombs.

8. A 100 microfarad condenser charged to a potential difference of 600 volts and is discharged through a resistance of 500 ohms. Find (1) the current when the current is decreasing at the rate of 16 amperes per second; (2) the charge on the condenser when the energy of the electrostatic field is 2 joules; (3) the potential drop through the condenser when the condenser is discharging at the rate of 0.5 coulomb per second; and (4) the rate at which the condenser is losing energy when the charge on the condenser is 10,000 microcoulombs.

Ans.: (1) 0.8 ampere; (2) 20,000 microcoulombs; (3) 250 volts; (4) 20 watts.

9. One-half a second after an *e. m. f.* of 1000 volts is im-

pressed upon a circuit formed by a non-inductive resistance and a condenser in series, the current is 10 milliamperes. If the resistance is 10,000 ohms, what is the capacity of the condenser?

Ans.: 21.7 microfarads.

10. An electrostatic voltmeter connected across the terminals of a condenser indicates a potential difference of 500 volts. The capacity of the condenser is 35 microfarads and that of the voltmeter is 5 microfarads. When an unknown resistance is connected across the condenser terminals, it is noted that when the condenser has been discharging for 1 second the voltage across its terminals has decreased to 100 volts. What is the value of the unknown resistance?

Ans.: 15,530 ohms.

11. Two-tenths of a second after an *e. m. f.* of 100 volts is impressed across the terminals of a circuit formed by a non-inductive resistance of 5000 ohms in series with a 20 microfarad condenser, a 30 microfarad condenser is connected in parallel with the 20 microfarad condenser. Find the charge on the 30 microfarad condenser 0.3 second after this parallel connection is made.

Ans.: 2410 microcoulombs.

12. A 50 microfarad condenser is charged to a potential difference of 1000 volts and is then discharged through a coil of 0.3 henry inductance and negligible resistance. Find (1) the maximum value of the current in the circuit; and (2) the frequency at which it oscillates. (3) Plot to scale the current and *p.d.* waves.

Ans.: (1) 12.91 amperes; (2) 41.1 cycles per second.

VII

ALTERNATING CURRENTS

165. Introduction. — We have just seen that when a condenser is discharged through a coil having an inductance L *but no resistance* that the current in the circuit and the *p.d.* across the condenser vary harmonically, *i.e.*, the instantaneous value of the current at any instant is $i = I_0 \sin \omega t$ and the instantaneous value of the *p.d.* across the condenser at any instant is $v = V_0 \cos \omega t$ where I_0 and V_0 are the maximum values of the current and *p.d.* and ω is equal to the factor 2π divided by the period of oscillation of the current and *p.d.* Both the current and *p.d.* vary periodically from fixed maximum values (I_0 and V_0 respectively) in one direction to equal maximum values ($-I_0$ and $-V_0$ respectively) in the other direction and back again to the maximum values in the first direction. The current is therefore called an *alternating* current and the *p.d.* an *alternating p.d.* This ideal circuit we have been considering is a very special one; alternating currents employed in practical work are produced by an entirely different method, namely, by the rotation of one or more coils in a magnetic field. The machine for producing alternating currents in this manner is called an *alternator*.

166. The Simple Alternator. — The simplest form of alternator consists of a coil of wire of one turn rotating in a uniform magnetic

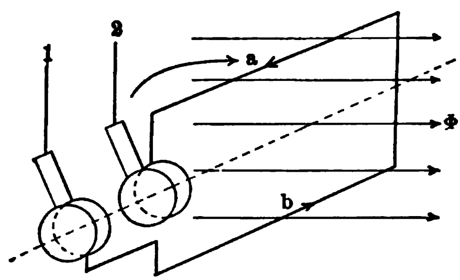


Fig. 94.

field, see Fig. 94. The ends of the wire forming the coil are connected to two rings called slip rings; making contact with

these rings are the brushes 1 and 2. Let the coil be rotated with an angular velocity ω; let time be counted from the instant when the wire a is above b and the plane of the coil coincides with a vertical plane drawn through the axis of rotation, and let ϕ_o be the value of the magnetic flux of induction threading the coil when in this position. Then $\phi = \phi_o \cos \omega t$ is the value of the flux threading the coil at any instant t; hence the *e. m. f.* induced in the coil at this instant is; see Article 105,

$$e = -\frac{d\phi}{dt} = \omega \phi_o \sin \omega t$$

and is in the direction from 1 to 2. The maximum value of the *e. m. f.* is then $\omega \phi_o$ and occurs when $\omega t = \frac{\pi}{2}$ or $t = \frac{\pi}{2\omega}$. Calling E_o this maximum value of the *e. m. f.*, the induced *e. m. f.* in the direction from 1 to 2 may be written

$$e = E_o \sin \omega t$$

The electromotive force in this case is a *harmonic* or *sine* function of the time, the period is $T = \frac{2\pi}{\omega}$ and the number of complete periods per second, or the *frequency*, is $f = \frac{\omega}{2\pi}$.

In alternators as actually constructed the coils of wire are not made of a single turn, nor do they rotate in uniform magnetic fields; the field of the alternator also has a large number of poles. By properly designing the coils and the pole faces of such a machine it is however possible to produce in the coils when rotated at constant speed an alternating *e. m. f.* which is practically a harmonic function of the time. In any case, the *e. m. f.* will be a periodic function of the time and can therefore be represented by a series of harmonic terms, called a *Fourier's Series*, of the form

$$e = E_1 \sin \omega t + E_2 \sin (2\omega t + \theta_2) + E_3 \sin (3\omega t + \theta_3) + - - - -$$

where ω depends solely upon the number of pairs of field poles and the angular velocity of the rotating part of the machine (which rotating part may be either the field or the armature) and the E's and θ's are constants, in general different for each term. The first term $E_1 \sin \omega t$ is called the *fundamental* or *first harmonic*, the remaining terms are called the second, third, etc., *harmonics*. The frequency of the *nth* harmonic is equal to n times the fre-

quency of the fundamental. In most practical cases it is necessary to consider only the first harmonic or fundamental.

In order to understand the properties of alternating electromotive forces and currents it is necessary first to get clearly in mind certain fundamental definitions.

167. Definition of Alternating Current and Alternating Electromotive Force. — An alternating current is defined as a current which varies continuously with time from a constant maximum value in one direction to an equal maximum value in the opposite direction and back again to the same maximum in the first direction, repeating this cycle of values over and over again in *equal*

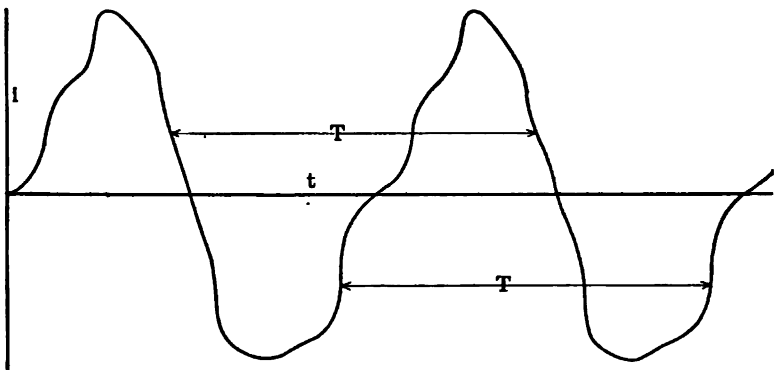


Fig. 95.

intervals of time T, in such a manner that the instantaneous value of the current at any instant t is identically the same as at any other instant $t+kT$ where T is the time, constant in value, required for the current to pass through a complete cycle of values, and k is any integer, positive or negative. Similarly, an alternating *e. m. f.* is defined as an *e. m. f.* which varies continuously with time from a constant maximum value in one direction to an equal maximum value in the opposite direction and back again to the same maximum in the first direction, repeating this cycle over and over again in *equal* intervals of time T, in such a manner that the instantaneous value of the *e. m. f.* at any instant t is identically the same as at any other instant $t+kT$ where T is the time, constant in value, required for the current to pass through a complete cycle of values, and k is any integer, positive or negative. In Fig. 95 the successive values of an alternating current are plotted as ordinates against time as the abscissa. Such a

curve is called a current "wave." In the figure the positive and negative portions of the current wave are shown unsymmetrical; such a non-symmetrical current or *p.d.* wave is physically possible, but the current and *p.d.* waves developed in ordinary electric machines are usually perfectly symmetrical, *i.e.*, the positive and negative portions of the wave are exactly alike.

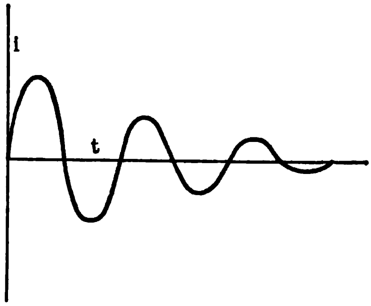


Fig. 96.

An *oscillating* current or *e.m.f.* is a current or *e. m. f.* which alternates in direction but changes in amplitude. An oscillating current wave is shown in Fig. 96.

A pulsating current is a current which varies with time but is always in the same direction; such currents are obtained from an arc rectifier.

163. Period, Frequency, Alternations, Periodicity and Phase. — To avoid repetition the following definitions are given in terms of an alternating current; they also apply to an alternating electromotive force, an alternating potential difference, or to any other periodic function of time.

The *period* of an alternating current is the time taken for the current to pass through a complete cycle of positive and negative values; *i.e.*, the period is equal to the time T defined in the preceding article.

The *frequency*, or *number of cycles per second*, is the number of periods per second.

The *number of alternations per minute* is the total number of times per minute that the current changes in direction, from positive to negative and from negative to positive. In engineering practice the number of cycles is usually referred to the second as the unit of time and the number of alternations is referred to the minute as the unit of time.

Let T be the period, f the frequency or number of cycles per second, and a the number of alternations per minute, then

$$f=\frac{1}{T} \qquad (1)$$

$$a=120f=\frac{120}{T} \qquad (2)$$

The equation of a harmonic or sine-wave current is

$$i = I_0 \sin (\omega t + \theta) \tag{3}$$

where ω is a constant equal to $\frac{2\pi}{T}$ and θ is a constant such that $I_0 \sin \theta$ gives the value of the current at time $t=0$. The constant ω is called the *periodicity* of the current, and the constant θ is called the *phase* of the current. The relation between periodicity, period and frequency is

$$\omega = \frac{2\pi}{T} = 2\pi f \tag{4}$$

169. Difference in Phase. — In general, when a harmonic or sine-wave electromotive force is impressed on a circuit the resulting current is likewise a harmonic function of time (after a very brief interval) having the same frequency, but the *e. m. f.* and current do not reach their maximum values simultaneously. In other words, when $e = E_0 \sin \omega t$ represents the *e. m. f.*, the current is represented by $i = I_0 \sin (\omega t + \theta)$ where E_0 and I_0 are the maximum values of the *e. m. f.* and current respectively; $\omega = 2\pi f$ where f is the frequency; t is the time measured from the instant when $e=0$ and is increasing in the *positive* direction and θ is an angle which measures the interval between the instants when the *e. m. f.* and current reach successive maximum values. The *e. m. f.* reaches its maximum value at time $t = \frac{\pi}{2\omega}$, while the current reaches its maximum value when

$$\omega t + \theta = \frac{\pi}{2} \text{ or } t = \frac{1}{\omega}\left(\frac{\pi}{2} - \theta\right)$$

Hence when θ is positive the current reaches its maximum value $\frac{\theta}{\omega}$ seconds *before* the *e.m.f.* reaches its maximum; when θ is negative the current reaches its maximum value $\frac{\theta}{\omega}$ seconds *after* the *e. m. f.* reaches its maximum. In the first case the current is said to *lead* the *e. m. f.*, and in the second case the current is said to *lag behind* the *e. m. f.* The angle θ is called the *difference in phase* between the current and *e. m. f.* A careful re-reading of the latter part of Article 23 will assist the reader in obtaining a clear physical conception of phase difference.

When the phase difference is zero the current and *e. m. f.* are

said to be *in phase;* when the phase difference is $\frac{\pi}{2}$ radians or 90° the current and *e. m. f.* are said to be *in quadrature;* when the phase difference is π radians or 180° the current and *e.m.f.* are said to be *in opposition.*

In general, when we have any two harmonic functions of time and of the same frequency $\frac{\omega}{2\pi}$, such as

$$x_1 = X_1 \sin(\omega t + \theta_1)$$

and

$$x_2 = X_2 \sin(\omega t + \theta_2)$$

the first function reaches its maximum value at time $t_1 = \frac{1}{\omega}\left(\frac{\pi}{2} - \theta_1\right)$ and the second reaches its first maximum value for $t_2 = \frac{1}{\omega}\left(\frac{\pi}{2} - \theta_2\right)$. Hence the first function x_1 reaches its first maximum an interval of time $t_2 - t_1 = \frac{1}{\omega}(\theta_1 - \theta_2)$ *ahead* of the second function x_2; the first function x_1 is therefore said to *lead* the second function x_2 by the angle $\theta_1 - \theta_2$. Note the order of the subscripts: x_1 *leads* x_2 by $\theta_1 - \theta_2$; or x_2 *leads* x_1 by the angle $\theta_2 - \theta_1$. A negative lead is of course equivalent to an actual lag, and a negative lag is equivalent to an actual lead.

170. Instantaneous, Maximum, and Average Values. — The *instantaneous* value of an alternating current is the value of the current at any instant.

The *maximum* value of an alternating current is the greatest instantaneous value during any cycle. For a harmonic alternating current the value of the current at any instant is given by an equation of the form $i = I_0 \sin(\omega t + \theta)$; in this case the constant I_0 is the maximum value.

The *average* value of an alternating current is defined as the numerical value of the average of its instantaneous values between successive zero values; hence, when the instantaneous values are plotted as ordinates against time as abscissa, the average value is the average ordinate for any positive half cycle of instantaneous values. In the case of a harmonic current of the form $i = I_0 \sin(\omega t + \theta)$ the average value $I_{aver.}$ is equal to 2 over π times the maximum value I_0, that is,

$$I_{aver.} = \frac{2}{\pi} I_o \tag{5}$$

For, put $\omega t + \theta = x$, then the average value of E_o *sin* $(\omega t + \theta)$ between the limits $\omega t + \theta = 0$ and $\omega t + \theta = \pi$ is equal to the average value of I_o *sin* x between the limits $x = 0$ and $x = \pi$, or

$$I_{aver.} = \frac{1}{\pi}\int_o^{\pi} I_o \sin x \, dx = \frac{I_o}{\pi}\Big[-\cos x\Big]_o^{\pi} = \frac{2}{\pi} I_o$$

The average value over a *complete* period of an alternating current having symmetrical positive and negative values is *zero,* since the average of the positive instantaneous values over half a period is equal to the average of the negative instantaneous values over half a period.

The above definitions and deductions also apply to alternating electromotive forces and alternating potential differences. In the case of any harmonic function of the form $x = X_o \sin(\omega t + \theta)$ the same relation exists between its maximum and average value as between the maximum and average value of a harmonic current, that is,

$$X_{aver.} = \frac{2}{\pi} X_o \tag{5a}$$

171. Effective Values. — The total amount of heat energy developed during a complete period T in a resistance r through which an alternating current is flowing is equal to

$$\int_o^T r\, i^2\, dt = r\int_o^T i^2 dt$$

A steady current I in the same interval of time will develop in this same resistance an amount of heat energy equal to rI^2T. Hence the alternating current will develop in any given resistance in any given time the same amount of heat energy as a direct current I provided

$$rI^2T = r\int_o^T i^2 dt$$

or

$$I^2 = \frac{1}{T}\int_o^T i^2 dt \tag{6}$$

The right-hand side of this equation represents the mean of the squares of the instantaneous values of the alternating current;

hence an alternating current will develop the same amount of heat energy in a given resistance as a direct current which has a value equal to the square root of the mean of the squares of the instantaneous values of the alternating current. *The square root of the mean of the squares of the instantaneous values of an alternating current over a complete period is called the* **effective** *value of the alternating current.* In specifying the value of an alternating current as so many amperes this effective value is always meant unless specifically stated otherwise. In the same manner *the square root of the mean of the squares of the instantaneous values of an alternating potential difference over a complete period is called the* **effective** *value of the alternating potential difference.* When the value of an alternating *potential difference* is specified as so many volts, this effective value is always meant unless specifically stated otherwise.

The reason for selecting this particular function of the instantaneous values of an alternating current or potential difference as the measure of the current or potential difference is that the deflection of all instruments used in alternating current measurements is a function of this effective value. Moreover, Joule's Law for the heating effect of a steady current also applies directly to the heating effect of an alternating current provided the current is expressed in terms of its effective value; *i.e.*, the average power dissipated in a resistance r, when an alternating current of *effective* value I flows through it, is rI^2.

In case the current is a harmonic function of time, a simple relation exists between its effective and maximum values. Let T be the period of the current and begin counting time when the current is zero and increasing in the positive direction. The equation of the current is then

$$i = I_0 \sin \omega t$$

where I_0 is its maximum value and $\omega = \frac{2\pi}{T}$. The effective value of the current is then

$$I = \sqrt{\frac{1}{T}\int_0^T I_0^2 \sin^2 \omega t \, dt}$$

But from the trigonometric relation that $\sin^2 x = \frac{1 - \cos 2x}{2}$, where x is any variable, we have that

$$\sin^2 \omega t = \frac{1}{2} - \frac{\cos 2\,\omega t}{2}$$

and therefore

$$\int_0^T \sin^2 \omega t\,dt = \left(\frac{t}{2} - \frac{\sin 2\,\omega t}{4\,\omega}\right)_{t=0}^{t=T} = \frac{T}{2}$$

Hence the effective value of the current is

$$I = \frac{I_0}{\sqrt{2}} \tag{6a}$$

Similarly the effective value of a harmonic electromotive force which has the maximum value E_0 is

$$E = \frac{E_0}{\sqrt{2}} \tag{6b}$$

and the effective value of a harmonic potential drop which has the maximum value V_0 is

$$V = \frac{V_0}{\sqrt{2}} \tag{6c}$$

Note that the effective value in each case is independent of the frequency but depends only upon the maximum value. *The above relations hold only for harmonic functions;* when the current, electromotive force or potential drop is not a harmonic function of the time equations (6) do not hold. (See Article 176.)

172. The Use of Alternating Currents. — When an alternating electromotive force is impressed across the terminals of an electric circuit of any kind an alternating current of the same frequency is established in this circuit.* When the strength of the current between these two points at any instant is i and the potential drop in the direction of the current is v, the input of electric energy between any two points 1 and 2 of the circuit in any infinitesimal

*The current established for the first second or two after the alternating electromotive force is impressed is not strictly an *alternating* current, since its amplitude builds up gradually to a constant maximum, just as the value of the current established in a circuit by a continuous electromotive force builds up gradually from zero to a constant maximum, depending upon the resistance and back electromotive forces in the circuit. The current for the first second or two is then an *oscillating* current. The building up of an alternating current is discussed in Article 201. For the present we shall confine our attention to what takes place in the circuit *after* the current has become a true alternating current, alternating between constant positive and negative maximum values.

interval of time dt is equal to $vidt$, and therefore the rate at which energy is transferred to this part of the circuit at this instant, or the *power input* between 1 and 2 at this instant, is $p = vi$. Although the average values of an alternating current and an alternating *p.d.* over a complete period are both zero, the average value of the power input (or output, when the potential drop is in the opposite direction to the current) when the current and *p.d.* have the same frequency is *not zero* except in certain special cases. Hence an alternating current may be used to transmit electric energy just as a continuous current is used for this purpose. In fact, the transmission of electric energy over long distances can be accomplished much more economically by the use of alternating than by the use of continuous currents. Alternating current generators are much less expensive for the same power output, and certain forms of alternating current motors, particularly the induction motor, are cheaper to build and require less care in operation than a direct current motor of the same power output. An additional advantage of alternating currents comes from the fact that by means of a device called an alternating current transformer, power may be readily transferred from a circuit in which the current is small and the voltage high to a circuit in which the voltage is low and the current large. Since the power lost in a transmission line depends upon the *square* of the current, it is obvious that for economical transmission the current should be kept small, and therefore the voltage high. On the other hand, a high voltage is dangerous, particularly inside of buildings. Hence electric power is usually generated at a comparatively low voltage, "stepped up" by means of a transformer to a high voltage for transmission, and then "stepped down" by means of another transformer to a comparatively low voltage for local distribution and use.

173. Alternating Current Transformer. — An alternating current transformer consists essentially of two independent windings wound on the same closed iron core. When a current is established in either winding a magnetic flux is established through the iron core, and when this current varies with time the flux varies with time and therefore an electromotive force is induced in each winding. Let N_1 and N_2 be the number of turns in series in the two windings respectively, and let ϕ be the number of lines of magnetic induction established in the core at any instant.

When all the lines of induction link all the turns of both windings (that is, when there is no magnetic leakage) the electromotive forces induced in the two windings at this instant are $e_1 = N_1 \frac{d\phi}{dt}$ and $e_2 = N_2 \frac{d\phi}{dt}$ respectively. Hence these electromotive forces are directly proportional to the number of turns in series in the two windings respectively. When the terminals of the first or primary winding are connected to the terminals of an alternating current generator and the terminals of the second or secondary winding are connected to an alternating current motor or other receiving device, energy will be transmitted through the transformer to the receiving device. Neglecting the dissipation of heat energy in the connecting wires and in the windings and core of the transformer, the power input into the primary winding of the transformer is equal to the power output of the secondary winding. Hence calling i_1 and i_2 the instantaneous values of the currents in the two windings, we have that

$$e_1 i_1 = e_2 i_2$$

But $\frac{e_1}{e_2} = \frac{N_1}{N_2}$; therefore $\frac{i_1}{i_2} = \frac{N_2}{N_1}$. That is, the electromotive forces induced in the two windings are to each other as the number of turns in the respective windings, and the currents in the two windings are to each other inversely as the number of turns in the respective windings. These relations are only approximate, since the assumed conditions are only approximately realized in practice. For a full discussion of the alternating current transformer see any text-book on alternating current machinery.

174. Average Power Corresponding to a Harmonic P.D. and a Harmonic Current. — Power-Factor. — The *average* power input into a circuit when a harmonic current is established in the circuit and a harmonic *p.d.* is established between the terminals of the circuit can be readily expressed in terms of the effective values of the current and the *p.d.* and the difference in phase between the current and *p.d.* Let

$$i = I_0 \sin (\omega t + \theta_1)$$

be the value of the current at any instant t; and let

$$v = V_0 \sin (\omega t + \theta_2)$$

be the value of the potential drop *in the direction of the current* i

at this same instant. The power input into the circuit at this instant is then

$$p = vi = V_0 I_0 \sin(\omega t + \theta_1) \sin(\omega t + \theta_2)$$

From the trigonometric relations that

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

we have, subtracting the upper from the lower, that

$$\sin a \sin b = \tfrac{1}{2}\cos(a-b) - \tfrac{1}{2}\cos(a+b) \qquad (7)$$

Hence, substituting $\omega t + \theta_1$ for a and $\omega t + \theta_2$ for b we have

$$\sin(\omega t + \theta_1)\sin(\omega t + \theta_2) = \tfrac{1}{2}\cos(\theta_1 - \theta_2) - \tfrac{1}{2}\cos(2\omega t + \theta_1 + \theta_2)$$

Hence the instantaneous power is

$$p = vi = \frac{V_0 I_0}{2}\cos(\theta_1 - \theta_2) - \frac{V_0 I_0}{2}\cos(2\omega t + \theta_1 + \theta_2) \qquad (8)$$

Therefore the instantaneous power is also a *harmonic* function of time but has *twice* the frequency of the current or *p.d.*, and it is unsymmetrical with respect to the axis of time unless $\theta_1 - \theta_2 \doteq \frac{\pi}{2}$, *i.e.*, unless the current and *p.d.* are in *quadrature*. The curves

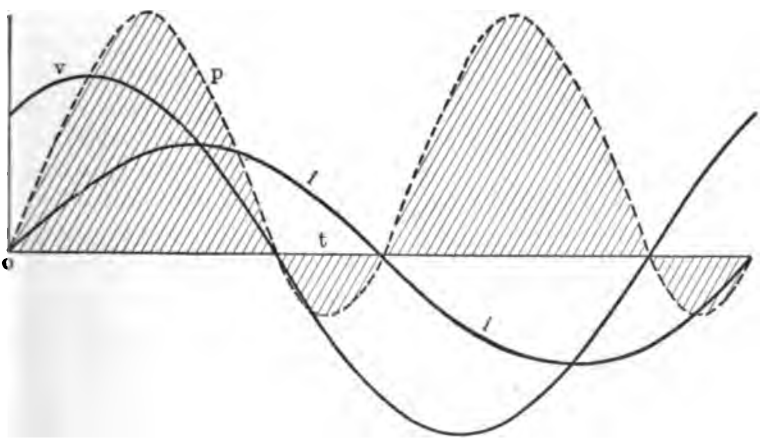


Fig. 97.

for current, *p.d.* and power in the general case are shown in Fig. 97. Note that, in general, for part of the time the power is positive and for part of the time the power is negative; which means that energy is transferred *to* the circuit during part of each cycle and transferred *from* the circuit during the remainder of the cycle.

The average power transferred to the circuit during each complete period T is

$$P=\frac{1}{T}\int_0^T vi\,dt=\frac{1}{T}\int_0^T\frac{V_oI_o}{2}\Big(\cos\,(\theta_1-\theta_2)-\cos\,(2\omega t+\theta_1+\theta_2)\Big)dt$$

Therefore

$$P=\frac{V_oI_o}{2T}\Big(t\cos\,(\theta_1-\theta_2)-\frac{1}{2\omega}\sin\,(2\omega t+\theta_1+\theta_2)\Big)_0^T$$

Whence

$$P=\frac{V_o\cdot I_o}{2}\cos\,(\theta_1-\theta_2)$$

and therefore the average power *input* is

$$P=VI\cos\,(\theta_1-\theta_2) \qquad (9)$$

where V and I are the effective values respectively of the *p.d.* and current, that is $V=\frac{V_o}{\sqrt{2}}$ and $I=\frac{I_o}{\sqrt{2}}$; and $\theta_1-\theta_2$ is the difference in phase between the current and *p.d.*

Hence, when the current and *p.d.* differ in phase, the average power input into the circuit is less than the product of the effective current and the effective *p.d.* The ratio of the true power input into a circuit to the product of the effective current and effective *p.d.* is called the *power factor* of the circuit. In the case of a harmonic current and a harmonic *p.d.* the power factor is therefore equal to the cosine of the angle which expresses the difference in phase between the current and the *p.d.* Hence this angle is frequently called the *power-factor angle* of the circuit.

Equation (9) may then be expressed in words as " the average power input into any circuit, when there is established a *harmonic* current in the circuit and a *harmonic p.d.* of the same frequency across the terminals of the circuit, is equal to the product of the effective values of the current and the *p.d.* times the power factor of the circuit."

When current and *p.d.* are in quadrature, *i.e.*, when $\theta_1-\theta_2=\pm\frac{\pi}{2}$ then $\cos\,(\theta_1-\theta_2)=0$, that is, the power factor is zero, and the average power input into the circuit is also zero. The sine curve representing the instantaneous power is then symmetrical with respect to the axis of time. In this case as much work is done *on* the current during one half of the cycle of the power curve as

is done *by* the current during the other half of this cycle; the total work done *by* the current in a whole cycle is zero. When current and *p.d.* are in phase, *i.e.*, when $\theta_1 - \theta_2 = 0$ then $\cos(\theta_1 - \theta_2) = 1$; that is, the power factor is unity and the average electric power input for the given current and *p.d.* is a maximum and equal to VI, where V and I are the effective values of *p.d.* and current. In this case the entire curve of instantaneous power lies above the axis of time. When current and *p.d.* are in opposition, *i.e.*, when $\theta_1 - \theta_2 = \pm \pi$ then $\cos(\theta_1 - \theta_2) = -1$; that is, the power factor is again numerically equal to unity, but the average electric power input is *negative*, *i.e.*, work is done *on* the current at the average rate VI. In this case the entire curve for instantaneous power lies *below* the axis of time.

In general, whenever the power-factor angle $\theta_1 - \theta_2$ is greater than $+\frac{\pi}{2}$ or less than $-\frac{\pi}{2}$, that is, whenever the current leads or lags behind the potential drop *in the direction of the current* by more than 90°, work is done on the current at the average rate equal to the numerical value of the expression $VI \cos(\theta_1 - \theta_2)$. This part of the circuit then acts like a generator. Let the instantaneous values of the net rise of potential in the direction of the current in this part of the circuit, *i.e.*, the terminal *e. m. f.*, be represented by the equation $e = E_0 \sin(\omega t + \theta')$. Then E_0 is numerically equal to V_0 and θ' is equal to $\theta_2 + \pi$, for the *rise* of potential at any instant is opposite to the *drop* of potential at this instant. The average power *output* of this part of the circuit is then

$$P_0 = EI \cos(\theta_1 - \theta') \qquad (9a)$$

where $E = V$ is the effective value of the terminal electromotive force. This *output* P_0 is positive when $(\theta_1 - \theta')$ is greater than $-\frac{\pi}{2}$ and less than $\frac{\pi}{2}$. Equation (9) gives the average electric power *input* into the circuit and equation (9*a*) gives the average electric power *output* of the circuit.

175. Power Corresponding to a Non-Harmonic P.D. and a Non-Harmonic Current. — When the *p.d.* and current are not harmonic functions of time the above expressions for power, equations (9), do not apply. An alternating *p.d.* or current, however, may always be represented by a Fourier's series (see Article 166), no

matter what may be the shape of the curve or "wave" representing it. For example, let the current and *p.d.* be

$$i = I_1 \sin(\omega t + \theta_1) + I_2 \sin(2\omega t + \theta_2) + I_3 \sin(3\omega t + \theta_3)$$

$$v = V_1 \sin(\omega t + \theta'_1) + V_2 \sin(2\omega t + \theta'_2) + V_3 \sin(3\omega t + \theta'_3)$$

In this case the instantaneous power is (see equation 7)

$$\begin{aligned} vi = \tfrac{1}{2}\big[& V_1 I_1 \cos(\theta_1 - \theta'_1) + V_2 I_2 \cos(\theta_2 - \theta'_2) + V_3 I_3 \cos(\theta_3 - \theta'_3) \\ & - V_1 I_1 \cos(2\omega t + \theta_1 + \theta'_1) - V_2 I_2 \cos(4\,\omega t + \theta_2 + \theta'_2) \\ & - V_3 I_3 \cos(6\omega t + \theta_3 + \theta'_3) + V_2 I_1 \cos(\omega t + \theta'_2 - \theta_1) \\ & + V_3 I_1 \cos(2\omega t + \theta'_3 - \theta_1) + V_1 I_2 \cos(\omega t + \theta_2 - \theta'_1) \\ & + V_3 I_2 \cos(\omega t + \theta_3' - \theta_2) + V_1 I_3 \cos(2\omega t + \theta_3 - \theta'_1) \\ & + V_2 I_3 \cos(\omega t + \theta_3 - \theta'_2) - V_2 I_1 \cos(3\omega t + \theta_1 + \theta'_2) \\ & - V_3 I_1 \cos(4\omega t + \theta_1 + \theta'_3) - V_1 I_2 \cos(3\omega t + \theta_2 + \theta_1') \\ & - V_3 I_2 \cos(5\omega t + \theta_2 + \theta'_3) - V_1 I_3 \cos(4\omega t + \theta_3 + \theta'_1) \\ & - V_2 I_3 \cos(5\omega t + \theta_3 + \theta'_2)\big] \end{aligned}$$

The average of this instantaneous power for a complete cycle of the fundamental period $T = \frac{2\pi}{\omega}$ is

$$P = \frac{1}{T}\int_0^T vi\,dt$$

Hence the average power is

$$P = \frac{V_1 I_1}{2}\cos(\theta_1 - \theta'_1) + \frac{V_2 I_2}{2}\cos(\theta_2 - \theta_2') + \frac{V_3 I_3}{2}\cos(\theta_3 - \theta_3') \quad (10)$$

since the integral between the limits 0 and T for each of the harmonic terms containing t in the equation for instantaneous power is zero. Hence when there exists in a circuit an alternating current and an alternating *p.d.* of any kind whatever, the average power is equal to the sum of the values of the average power corresponding to each pair of harmonics of the *same* frequency which exist in the Fourier's series for the current and *p.d.* That is, the average power input corresponding to each pair of harmonics of the *same* frequency is independent of what other harmonics may be present. Note that when any harmonic is absent in *either* the *p.d.* or the current this harmonic contributes nothing to the average power. For example the term $\frac{I_3 V_3}{2}\cos(\theta_3 - \theta'_3)$ equals zero for either $I_3 = 0$ or $V_3 = 0$.

It should also be remembered that in most practical cases the curves representing *p.d.* and current are each symmetrical with

respect to the axis of time, and therefore as a rule only the odd harmonics are present.

176. Effective Value of a Non-Harmonic Current or P.D. — When the current is of the form

$$i = I_1 \sin(\omega t + \theta_1) + I_2 \sin(2\omega t + \theta_2) + I_3 (\sin 3\omega t + \theta_3)$$

the square of the effective value is, by definition, $I^2 = \frac{1}{T}\int_0^T i^2 dt.$

That is, the square of the effective value of the current is of the same mathematical form as the average power corresponding to the current i and an equal *p.d.* in phase with it. Hence from equation (10)

$$I^2 = \frac{I_1^2}{2} + \frac{I_2^2}{2} + \frac{I_3^2}{2} + \text{——}$$

But $\frac{I_1}{\sqrt{2}}, \frac{I_2}{\sqrt{2}}, \frac{I_3}{\sqrt{2}}$, etc., are the *effective* values of the harmonics which have the maximum values I_1, I_2, I_3, etc., respectively. Hence, the *effective value of any non-harmonic current is equal to the square root of the sum of the squares of the effective values of all the harmonics present in the current wave.* Similarly, the effective value of any non-harmonic *p.d.* is equal to the square root of the sum of the squares of the effective values of all the harmonics present in the *p.d.* wave. When I_1, I_2, I_3, etc., are taken to represent the *effective* values of the various harmonics of the current and V_1, V_2, V_3, etc., are taken to represent the *effective* values of the various harmonics of the *p.d.*, the effective value of the resultant current may be written

$$I = \sqrt{I_1^2 + I_2^2 + I_3^2 + \text{——}} \qquad (11)$$

and the effective value of the *p.d.* may be written

$$V = \sqrt{V_1^2 + V_2^2 + V_3^2 + \text{——}} \qquad (11a)$$

and the average power may be written

$$P = I_1 V_1 \cos\theta_1 + I_2 V_2 \cos\theta_2 + I_3 V_3 \cos\theta_3 + \text{——} \qquad (11b)$$

where θ_1, θ_2, θ_3, etc., are respectively the *differences* in phase between the harmonics of current and *p.d.* of the *same* frequency.

177. Equivalent Sine-Wave P.D. and Current. — In practical work the *p.d.* and current are seldom simple harmonic functions of time, *i.e.*, are seldom "sine waves," but each contains one or more of the odd harmonics. As a rule, however, it is not necessary to consider these harmonics separately, but as a first approximation the *p.d.* and current may each be considered as a

sine wave having an effective value equal to the effective value of the actual wave and differing in phase by an angle θ such that $VI \cos\theta$ represents the average power, where V and I are the effective values of the true *p.d.* and current waves respectively. In certain special cases, however, it is necessary to analyse the waves into their constituent harmonics.

178. Determination of the Maximum Value and Phase of the Harmonics in a Wave of Any Shape. — When the wave shape of a *p.d.* or current is known, the effective value and the phase of the harmonic of any order may be readily determined. For example, the instantaneous value of the current may be written

$$i = I_1 \sin x + I_2 \sin (2x + \theta_2) + I_3 \sin (3x + \theta_3) + \text{etc.} \qquad (a)$$

where $x = 2\pi ft$ and f is the frequency of the wave. The instantaneous value of the current i corresponding to any value of x is also given by the corresponding ordinate of the curve representing the current wave. We wish to determine the constants I_1, I_2, I_3, etc., and θ_2, θ_3, etc., which will make the curve represented by (a) coincide with the actual current wave. Consider, for example, the third harmonic. Multiply equation (a) by $\sin 3x$ and integrate with respect to x over an entire period of the wave, *i.e.*, between the limits $x = o$ and $x = 2\pi$. We then have

$$\int_o^{2\pi} i \sin 3x \, dx = \int_o^{2\pi} [I_1 \sin x \sin 3x + I_2 \sin (2x + \theta_2) \sin 3x + I_3 \sin (3x + \theta_3) \sin 3x + \text{ etc.}] \, dx$$

But the integral of each term in the right-hand member of this equation between the limits o and 2π is zero, except for the particular term $I \sin (3x + \theta) \sin 3x$, the integral of which from 0 to 2π is $\frac{2\pi I_3}{2} \cos\theta_3 = \pi I_3 \cos\theta_3$. (See Article 175.) Hence

$$\int_o^{2\pi} i \sin 3x \, dx = \pi I_3 \cos \theta_3$$

The integral $\int_o^{2\pi} i \sin 3x \, dx$ may be determined graphically by plotting the expression $i \sin 3x$ or ordinates against x as abscissas, and determining the area of this curve by means of a planimeter. Call this area A_3, then

$$A_3 = \pi I_3 \cos \theta_3 \qquad (b)$$

Next, multiply the equation (a) by $\cos 3x = \sin\left(3x + \frac{\pi}{2}\right)$. In exactly the same manner we then have that

$$\int_0^{2\pi} i \cos 3x\, dx = \pi I_3 \cos\left(\frac{\pi}{2} - \theta_3\right) = \pi\, I_3 \sin \theta_3$$

and the value of $\int_0^{2\pi} i \cos 3x\, dx$ may be determined graphically as in the first case by plotting the curve $i \cos 3x$ and finding its area by means of a planimeter. Call this area B_3, then

$$B_3 = \pi I_3 \sin \theta_3 \qquad (c)$$

From equations (b) and (c) we then have

$$I_3 = \frac{1}{\pi}\sqrt{A_3^2 + B_3^2} \qquad (12a)$$

and

$$\theta_3 = \tan' \left(\frac{B_3}{A_3}\right) \qquad (12b)$$

Note that in case the wave is symmetrical with respect to the axis of time it is unnecessary to look for the *even* harmonics. Also, when the wave is symmetrical, the curves $i \sin 3x$ and $i \cos 3x$ need be plotted for only a *half* period of the wave. Calling α_3 and β_3 the areas of these curves for half a period of the wave, we then have

$$I_3 = \frac{2}{\pi}\sqrt{\alpha_3^2 + \beta_3^2} \qquad (13a)$$

$$\theta_3 = \tan^{-1}\left(\frac{\beta_3}{\alpha_3}\right). \qquad (13b)$$

This method of determining the harmonics present in the wave is of course applicable to the determination of the harmonic of any order.

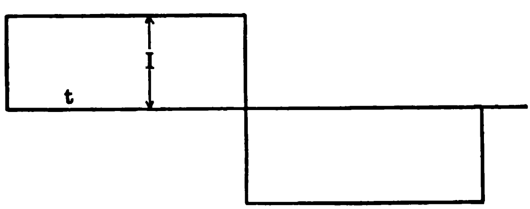


Fig. 98.

Example: Suppose the current is a symmetrical rectangular wave as shown in Fig. 98. Let I be the value of the current

during the positive half of the wave. Since the wave is symmetrical, only the odd harmonics can be present. Consider the nth harmonic, where n is any *odd* positive integer. Then

$$a_n = \int_0^\pi I \sin nx\, dx = \left(-\frac{I}{n} \cos nx\right)_0^\pi = \frac{2I}{n}$$

and

$$\beta_n = \int_0^\pi I \cos nx\, dx = \left(\frac{I}{n} \sin nx\right)_0^\pi = 0$$

since n is *odd*. Hence from equations (13)

$$I_n = \frac{2}{\pi}\sqrt{\left(\frac{2I}{n}\right)^2 + (0)^2} = \frac{4I}{\pi n}$$

and

$$\theta_n = \tan \frac{0}{\frac{2I}{n}} = 0$$

Hence all the *odd* harmonics exist in a symmetrical rectangular wave, the maximum values of the harmonics varying inversely as their order n. A symmetrical rectangular wave having a maximum value I may then be represented by the infinite series

$$i = \frac{4I}{\pi} \sum_{n=1}^{n=\infty} \frac{\sin (2\, \pi\, n\, ft.)}{n}$$

where n has all the *odd* values from 1 to ∞ and f is the frequency of the wave.

179. The Fisher-Hinnen Method for Analysing a Non-Harmonic Wave. —(See *Electric Journal*, Vol. 5, p. 386 and *Elektrotechnische Zeitschrift*, May 9, 1901.) This method is much simpler than that described above, except in the rare cases where the resultant wave may be represented by a simple integrable function. The method is based on the following facts:

1. The *algebraic* sum of any n equally spaced ordinates of a *sine* wave, when these ordinates are spaced $\frac{k}{n}$th of a wave length apart, where k is any integer which is *not* a multiple of n, is zero.

2. The algebraic sum of n ordinates of a *sine* wave when these ordinates are spaced $\frac{k}{n}$ wave lengths apart, *where k is a multiple of n*, is equal to n times the ordinate of this wave at any one of these points.

3. The maximum ordinate of any *sine* wave is equal to the square root of the sum of the squares of any two ordinates spaced a quarter of a wave length apart.

4. Let y_1 be the ordinate of a *sine* wave at any point at an angular distance x_1 from the origin, and let y_2 be another ordinate of this wave a quarter of a wave length to the *right* of x_1. Then the angular distance measured from the point x_1 toward the *left* to the point at which this wave first crosses the x axis in the positive direction is

$$\phi = tan^{-1}\frac{y_1}{y_2}$$

Consider a wave of any form whatever (for example, the wave shown in Fig. 95) and let the highest harmonic of this wave be the nth. Let $y_1, y_3, y_5,$ ------- y_{2n-1} be n ordinates of this wave spaced $\frac{360}{n}$ degrees apart, where 360° corresponds to a complete wave length of the given wave. Let a_1', a_3', a_5', be the corresponding ordinates of the fundamental or first harmonic of this wave, a_1'', a_3'', a_5'', be the corresponding ordinates of the second harmonic of this wave, a_1''', a_3''', a_5''', etc., be the corresponding ordinates of the third harmonic, and so on; the corresponding ordinates of the nth harmonic being $a_1^{(n)}$, $a_3^{(n)}$, $a_5^{(n)}$, etc. We then have that

$$y_1 = a_1' + a_1'' + a_1''' + \text{------------} a_1^{(n)}$$
$$y_3 = a_3' + a_3'' + a_3''' + \text{------------} a_3^{(n)}$$
$$y_5 = a_5' + a_5'' + a_5''' + \text{------------} a_5^{(n)}.$$
$$\text{--------------------------}$$
$$y_{2n-1} = a_{2n-1}' + a_{2n-1}'' + a_{2n-1}''' + \text{-------} a_{2\,n-1}^{(n)}$$

The ordinates a_1' to a_{2n-1}' of the first harmonic are ordinates of a sine wave and are spaced $\frac{1}{n}$th of a wave length apart, and therefore, from Proposition 1, their sum is zero. Similarly, the ordinates a_1'' to a_{2n-1}'' are ordinates of a *sine* wave of *half* the wave length of the fundamental and therefore the angular distance between these ordinates is $\frac{2}{n}$th of the wave length of the *sine* wave of which they are the ordinates; hence the sum of the ordinates a_1'' to a_{2n-1}'' is zero. Similarly for all the other ordinates except those of the nth harmonic. The latter are spaced $\frac{n}{n}=1$ wave length apart, and therefore, from Proposition 2, their sum is equal to n times the value of the ordinate of this harmonic at any one of the points 1, 3, 5, etc. Hence the value of the ordinate of the nth harmonic at the point 1 is

$$A_1 = \frac{1}{n}(y_1 + y_3 + y_5 + \text{-----------} y_{2n-1})$$

Similarly, if $y_2, y_4, y_6,$ --------- y_{2n} are ordinates of the given wave one quarter of a wave length of the nth harmonic to the

right of the first set of ordinates, their sum will be equal to n times the ordinate of the nth harmonic a quarter of a wave length of this harmonic from y_1; call this ordinate B_n, then

$$B_n = \frac{1}{n}(y_2 + y_4 + y_6 + \text{-----------} y_{2n})$$

Then from Proposition 3, the maximum value of the ordinate of the nth harmonic is

$$Y_n = \sqrt{A_n^2 + B_n^2}$$

From Proposition 4, the angular distance to the *left* of y_1 at which this harmonic cuts the x axis in the positive direction is

$$\phi'_n = tan^{-1}\frac{A_n}{B_n}$$

when 360 degrees are taken equivalent to a wave length of the nth harmonic. When 360 degrees are taken equivalent to a wave length of the *given wave* this angular distance is

$$\phi_n = \frac{1}{n}\, tan^{-1}\frac{A_n}{B_n}$$

Consider next the mth harmonic, and erect two sets of m ordinates, the ordinates of each set being spaced 360° apart (considering a wave length of the given wave as equivalent to 360°) and the second set a quarter of a wave length of this harmonic to the right of the first set. Then, if the harmonics of higher order are not multiples of m, we have as before that the ordinate of the mth harmonic at the point 1 is

$$A_m = \frac{1}{m}(y_1 + y_3 + y_5 \text{-----------} y_{2m-1})$$

and the ordinate of the mth harmonic at the point 2, which is a quarter wave length of this harmonic to the right of 1, is

$$B_m = \frac{1}{m}(y_2 + y_4 + y_6 + \text{-----------} y_{2m})$$

Whence the maximum value of this harmonic is

$$Y_m = \sqrt{A_m^2 + B_m^2}$$

and it cuts the x axis at the angular distance

$$\phi_m = \frac{1}{m}\, tan^{-1}\frac{A_m}{B_m}$$

to the *left* of the first ordinate when 360 degrees are taken equivalent to the wave length of the original wave.

If there also exists in the given wave a harmonic of the nth order, where n is a multiple of m, that is if $n = km$, where k is an integer, then from Proposition 2, since each set of these m ordinates is spaced $\frac{n}{m} = k$ wave lengths of the nth harmonic apart, we have that the sum of the first set of m ordinates also contains m times

the ordinate of the nth harmonic at the point 1. Calling A_n' the ordinate of the nth harmonic at the point 1 we then have that

$$A_m = \frac{1}{m}\,(y_1 + y_3 + y_5 + \text{-----------} + y_{2m-1}) - A_n'$$

Similarly,

$$B_m = \frac{1}{m}\,(y_2 + y_4 + y_6 + \text{-----------} y_{2m}) - B_n'$$

where B_n' is the ordinate of the nth harmonic at the point 2.

A similar correction must be applied for all other harmonics of higher order than the mth if the orders of these harmonics are multiples of m.

The waves of current and electromotive force with which one has to deal in practice usually contain only the odd harmonics; also, as a rule, the harmonics of higher orders than the seventh are negligible. In this case the three harmonics, the third, fifth, and seventh, are not multiples of each other and consequently no correction term has to be applied. Moreover, it is sufficient to consider the ordinates of only half a wave length. To determine the third harmonic divide the base line of this half wave into $2n = 6$ equal parts and measure the ordinates at the beginning of each of these six segments. Let these ordinates be $y_1, y_2,$ -------- y_6. Let the beginning of the first segment be taken where the given wave cuts the x axis and call this point the origin, then $y_1 = 0$ and we have

$$A_3 = \frac{1}{3}\,(y_5 - y_3)$$

$$B_3 = \frac{1}{3}\,(y_2 + y_6 - y_4)$$

Then the maximum value of the third harmonic is

$$Y_3 = \sqrt{A_3^2 + B_3^2}$$

and it cuts the x axis at the angular distance

$$\phi_3 = \frac{1}{3}\ tan^{-1}\left(\frac{A_3}{B_3}\right)$$

to the *left* of the origin. The equation of the third harmonic is then

$$y_3 = Y_3\, sin\, 3\,(x + \phi_3)$$

Similarly, starting at the same point and dividing the half wave into $2n = 10$ segments, we have for the 5th harmonic

$$A_5 = \frac{1}{5}\,(y_5 + y_9 - y_3 - y_7)$$

$$B_5 = \frac{1}{5}\,(y_2 + y_6 + y_{10} - y_4 - y_8)$$

$$Y_5 = \sqrt{A_5^2 + B_5^2}$$

$$\phi_5 = \frac{1}{5} \tan^{-1} \left(\frac{A_5}{B_5}\right)$$

and the equation of this harmonic is

$$y_5 = Y_5 \sin 5\,(x + \phi_5)$$

The determination of the seventh harmonic is carried out by an exactly similar process, dividing the base line into $2n = 14$ equal parts.

The ordinates of the fundamental at the origin and at the point corresponding to one quarter of a wave length from the origin are then respectively

$$A_1 = -A_3 - A_5 - A_7$$

and

$$B_1 = y_m + B_3 - B_5 + B_7$$

where y_m is the ordinate of the given wave corresponding to the point a quarter of a wave length from the origin (*i.e.*, the mid-ordinate of the given wave) and A_3, A_5, A_7, B_3, B_5, and B_7 are the quantities above determined. The maximum value of the fundamental is then

$$Y_1 = \sqrt{A_1^2 + B_1^2}$$

and it cuts the axis of x at the angular distance

$$\phi_1 = \tan^{-1} \left(\frac{A_1}{B_1}\right)$$

to the left of the origin. Its equation is therefore,

$$y_1 = Y_1 \sin (x + \phi_1)$$

The equation of the given wave is then

$$y = Y_1 \sin (x + \phi_1) + Y_3 \sin 3\,(x + \phi_3) + Y_5 \sin 5\,(x + \phi_5) + Y_7 \sin 7\,(x + \phi_7) \qquad (14a)$$

The effective value of the given wave is

$$Y = \sqrt{\frac{Y_1^2 + Y_3^2 + Y_5^2 + Y_7^2}{2}} \qquad (14b)$$

and the average value is

$$Y_{aver} = \frac{2}{\pi}\left[Y_1 \cos \phi_1 + \frac{1}{3} Y_3 \cos 3\,\phi_3 + \frac{1}{5} Y_5 \cos 5\phi_5 + \frac{1}{7} Y_7 \cos 7\phi_7\right] \qquad (14c)$$

In employing the above formulas strict attention must be paid to algebraic signs.

Example. In the curve shown in the figure the ordinates at 6 equally spaced points, starting from the point where the curve cuts the base line are

$y_1 = 0$	$y_4 = 940$
$y_2 = 676$	$y_5 = 1004$
$y_3 = 660$	$y_6 = 554$

and the ordinates corresponding to 10 equally spaced points, starting from the point where the curve cuts the base line are

$y_1=0$	$y_3=719$	$y_5=702$	$y_7=1086$	$y_9=639$
$y_2=470$	$y_4=678$	$y_6=940$	$y_8=920$	$y_{10}=375$

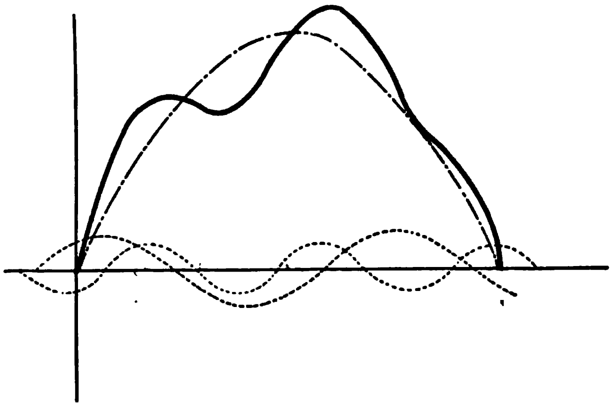


Fig. 99.

From the first set of ordinates we have for the third harmonic

$$A_3=\frac{1004-660}{3}=+114.7$$

$$B_3=\frac{676+554-940}{3}=+96.7$$

$$Y_3=\sqrt{(114.7)^2+(96.7)^2}=150$$

$$\phi_3=1\ tan^{-1}\frac{114.7}{96.7}=+16.6°$$

From the second set of ordinates we have for the fifth harmonic

$$A_5=\frac{702+630-719-1086}{5}=-92.8$$

$$B_5=\frac{470+940+375-678-920}{5}=+37.4$$

$$Y_5=\sqrt{(92.8)^2+(37.4)^2}=100$$

$$\phi_5=\frac{1}{5}\ tan^{-1}\frac{-92.8}{37.4}=-13.6°$$

For the fundamental we then have

$$A_1=114.7+92.8=-21.9$$

$$B_1=940+96.7-37.4=+999.3$$

$$Y_1=\sqrt{(21.9)^2+(999.3)^2}=999.5$$

$$\phi^1=tan^{-1}\frac{-21.9}{999.3}=-1.25°$$

The equation of the given wave is then

$$y=999.5 \sin (x-1.25°)+150 \sin 3 (x+16.6°) \\ +100 \sin 5 (x-13.6°)$$

Its effective value is

$$Y=718$$

and its average value

$$Y_{aver}=673$$

180. Form Factor. — The form factor of a wave is defined as the ratio of the effective value to the average value. For a sine wave the form factor is $\frac{I_0}{\sqrt{2}} \div \frac{2I_0}{\pi}=\frac{\pi}{2\sqrt{2}}=1.11$. The form factor of a flat-topped wave is less; of a peaked wave greater. The form factor of the wave shown in Fig. 99 is $\frac{718}{673}=$ 1.068.

181. Amplitude Factor. — The amplitude factor of a wave is defined as the ratio of the maximum value to the effective value. For a sine wave the amplitude factor is 1.414. The amplitude factor of the wave shown in Fig. 99 is 1.53.

182. Power and Reactive Components of P.D. and Current.— Whenever we have to deal with two or more harmonic functions of time, we may as a matter of convenience begin counting time at an instant when one of these functions is zero and is increasing in the positive direction. For example, when the current is $i=I_0 \sin (\omega t+\theta_1)$ and the *p.d.* $v=V_0 \sin (\omega t+\theta_2)$, we may begin counting time at the instant when $v=0$ and is increasing in the positive direction; θ_2 is then equal to zero. Hence, dropping the subscript from θ_1 and also writing $I_0=\sqrt{2}\, I$ and $V_0=\sqrt{2}\, V$, where I and V are the *effective* values of i and v respectively, we have

$$v=\sqrt{2}\, V \sin \omega t$$

$$i=\sqrt{2}\, I \sin (\omega t+\theta)$$

The average power is then

$$P=VI \cos \theta \tag{15}$$

From the trigonometric formula that

$$\sin (a+b)=\sin a \cos b+\cos a \sin a$$

we may write the current

$$i=\sqrt{2}\, I \cos \theta \sin \omega t+\sqrt{2}\, I \sin \theta \cos \omega t$$

that is, we may consider i as made up of two components

$$i_1 = \sqrt{2} I \cos \theta \sin \omega t$$

$$i_2 = \sqrt{2} I \sin \theta \cos \omega t = \sqrt{2} I \sin \theta \sin \left(\omega t + \frac{\pi}{2}\right)$$

The first component i_1 is in phase with the *p.d.* and has the effective value $I \cos \theta$; the second component is in quadrature *ahead* of the *p.d.* and has the effective value $I \sin \theta$. The average power corresponding to the first component i_1 is

$$V (I \cos \theta) \cos 0 = VI \cos \theta = P$$

and the average power corresponding to the second component i_2 is

$$V (I \sin \theta) \cos \frac{\pi}{2} = 0$$

Hence the *average power of a harmonic p.d. and current is equal to the effective value of the p.d. times the effective value of the component of the current in phase with the p.d.* The component of the current in phase with the *p.d.* is therefore called the *power* component of the current. The component of the current in quadrature with the *p.d.* is called the *reactive* component of the current, since the power *input* corresponding to this component of the current during each quarter cycle is exactly equal to the power *given back* during the following quarter cycle.

Similarly, we may consider the *p.d.* as made up of two components

$$v_1 = \sqrt{2} V \cos \theta \sin (\omega t + \theta)$$

$$v_2 = \sqrt{2} V \sin \theta \cos (\omega t + \theta) = \sqrt{2} V \sin \theta \sin \left[(\omega t + \theta) - \frac{\pi}{2}\right]$$

The first component v_1 is in phase with the current and has the effective value $V \cos \theta$; the second component v_2 is in quadrature *behind* the current and has the effective value $V \sin \theta$. The average power corresponding to the first component v_1 is

$$I (V \cos \theta) \cos 0 = I V \cos \theta = P$$

and the average power corresponding to the second component v_2 is

$$I (V \sin \theta) \cos \frac{\pi}{2} = 0$$

Hence the average power of a harmonic p.d. and current is also equal to the product of the effective value of the current times the effective value of the component of the p.d. in phase with the current. The effective value of the component of the *p.d.* in phase with

the current is therefore called the *power* component of the *p.d.*; and the effective value of the component of the *p.d.* in quadrature with the current is called the reactive component of the *p.d.* The reactive component of the *p.d.* is in quadrature *behind* the current when the resultant current *leads* the resultant *p.d.* and is in quadrature *ahead of* the current when the resultant current *lags behind* the resultant *p.d.*

The name "wattless component" is sometimes used for reactive component, since the *average* watts corresponding to this component of current or *p.d.* is zero; the *instantaneous* watts corresponding to this component, however, are not zero. Hence the adjective "wattless" is misleading.

183. Apparent Power. — The product of the effective value V of the resultant *p.d.* and the effective value I of the resultant current is called the *apparent power*, that is

$$\text{Apparent Power} = VI \tag{16}$$

From equation (15), we have

$$\text{Average Power} = VI \cos \theta$$

where θ is the difference in phase between *p.d.* and current. Hence

$$\text{Power Factor} = \frac{\text{Average Power}}{\text{Apparent Power}} = \cos \theta$$

The terms *volt-amperes* and *apparent watts* are also frequently used for apparent power; *i.e.*, volt-amperes or apparent watts = (effective *p.d.* in volts) × (effective current in amperes).

184. Reactive Power. — The expression *reactive power* is used for the product of the effective current and the effective value of the *p.d.* in quadrature with it; or, what amounts to the same thing, the product of the effective *p.d.* and the effective value of the component of the current in quadrature with the *p.d.* That is

$$\text{Reactive Power} = VI \sin \theta \tag{17}$$

where V and I are the effective values of *p.d.* and current respectively, and θ is the power-factor angle. From equations (15), (16) and (17) it follows that the apparent power is equal to the square root of the sum of the squares of the average power and the reactive power, *i.e.*,

$$VI = \sqrt{(VI \cos \theta)^2 + (VI \sin \theta)^2}$$

185. Addition of Alternating Currents and of Alternating Potential Differences. — In any technical problem that has to do with the generation, distribution or utilisation of electric energy by means of alternating currents it is of fundamental importance to be able to predetermine the distribution of the currents and potential drops in the various windings of the machines and in the transmission line or network which forms the distribution system. The fundamental principles involved in such calculations are the same as in the case of continuous currents, that is, the two principles known as Kirchhoff's Laws. However, in the ease of alternating currents, Kirchhoff's Laws in the simple form in which they are stated in Article 98 *apply only to the instantaneous values of the currents and electromotive forces;* they do not apply to the effective values of these quantities. Effective values of alternating currents and electromotive forces are not additive. Effective values of alternating currents and electromotive forces are like the *numerical* values of continuous currents and electromotive forces; we cannot say, when two batteries are connected in a circuit, whether the net electromotive force in the circuit will be the sum or the difference of the electromotive forces of the two batteries *unless we know their direction with respect to each other.* Similarly, when two alternating electromotive forces are acting in the same circuit, we cannot say what will be the net electromotive force in the circuit unless we know their directions with respect to each other. The numerical value of an alternating electromotive force is specified by its effective value, and the difference in direction between two alternating electromotive forces is specified by their difference in phase. Hence, to *determine the effective value of two alternating electromotive forces, it is necessary to know both their effective values and their difference in phase.* Similarly, *to determine the effective value of the total current leaving any junction in a network of circuits, it is necessary to know not only the effective values of the currents coming up to that junction but to know also the phase relations of these currents.*

Consider first two harmonic alternating electromotive forces of the same frequency,

$$e_1 = E_1 \sin \omega t$$

and

$$e_2 = E_2 \sin (\omega t + \theta)$$

in series between any two points of a circuit. These two electro-

motive forces then differ in phase by $\theta°$. If $\theta = 0$, the two act in the same direction at each instant; if $\theta = 180°$ the two oppose each other at each instant; if θ has any other value the two electromotive forces act together during part of each cycle and oppose each other during the rest of the cycle. In any case, the net or resultant electromotive force at any instant is

$$e = e_1 + e_2 = E_1 \sin \omega t + E_2 \sin (\omega t + \theta)$$

The expression $E_1 \sin \omega t + E_2 \sin (\omega t + \theta)$ may be put equal to $E_o \sin (\omega t + \theta_o)$ where the constants E_o and θ_o may be determined from the condition that the relation

$$E_1 \sin \omega t + E_2 \sin (\omega t + \theta) = E_o \sin (\omega t + \theta_o)$$

must hold at all times. For $t = 0$, we then have that

$$E_2 \sin \theta = E_o \sin \theta_o \tag{a}$$

and for $t = \dfrac{\pi}{2\omega}$,

$$E_1 + E_2 \cos \theta = E_o \cos \theta_o \tag{b}$$

since $\sin \left(\dfrac{\pi}{2} + \theta\right) = \cos \theta$ and $\sin \left(\dfrac{\pi}{2} + \theta_o\right) = \cos \theta_o$

Squaring the equations (*a*) and (*b*) and adding, we get

$$E_1^2 + E_2^2 \cos^2 \theta + 2E_1E_2 \cos \theta + E_2^2 \sin^2 \theta = E_o^2 \cos^2 \theta_o + E_o^2 \sin^2 \theta_o$$

whence

$$E_o^2 = E_1^2 + E_2^2 + 2E_1E_2 \cos \theta \tag{18a}$$

Dividing the equation (*a*) by the equation (*b*), we get

$$\tan \theta_o = \frac{E_2 \sin \theta}{E_1 + E_2 \cos \theta} \tag{18b}$$

Therefore

$$e_1 + e_2 = E_o \sin (\omega t + \theta_o)$$

where E_o is the maximum value of the resultant *e.m.f.* and is given by equation (18*a*) and θ_o is the angle by which the resultant *e. m. f. leads* e_1 and is given by equation (18*b*).

Exactly similar relations hold for two harmonic currents of the same frequency, and in fact for any two harmonic functions of the same frequency in the same independent variable.

186. Representation of a Harmonic Function by a Rotating Vector. — The results just deduced may be arrived at graphically by a very simple method. Let OQ in Fig. 100 be any line fixed in the plane of the paper and let the line OP_1 be equal in length to E_1, and be pivoted at O and rotate about O with an angular

velocity ω; let this rotating line coincide with the line of reference OQ at time $t=0$. Then at any instant of time t, this line will make an angle ωt with OQ and the instantaneous value of the *e.m.f.* $e_1=E_1 \sin \omega t$ at any instant will be equal to the vertical distance from P_1 to OQ. Similarly, if OP_2 is a second line equal in length to E_2 rotating about O with the same angular velocity ω, and making the angle θ with OQ at time $t=0$, then the instantaneous value of the *e. m. f.* $e_2=E_2 \sin(\omega t+\theta)$ at any instant will be equal to the vertical distance from P_2 to OQ. Since both lines rotate with the same velocity the angle between the two will be equal to θ at all times; that is the phase angle θ measures the difference in direction between the two rotating vectors representing the two *e. m. f.*'s.

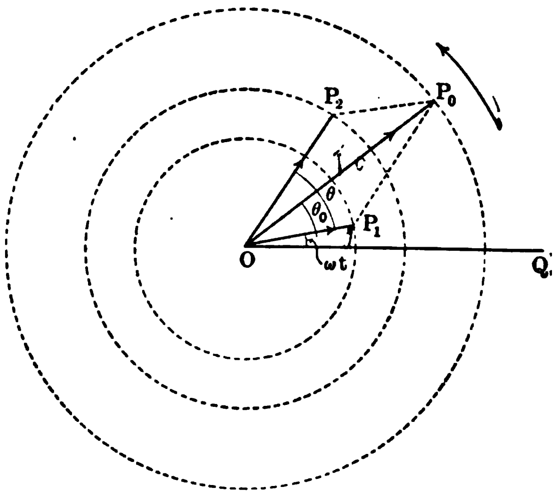


Fig. 100.

Let $OP_0=E_0$ be the vector sum of OP_1 and OP_2. It is then evident from the diagram that the instantaneous value of (e_1+e_2) at any instant is the vertical distance from P_0 to OQ; also that this line OP_0 remains fixed in length and fixed in position with respect to OP_1 and OP_2 and is equal numerically to the maximum value of e_1+e_2, and at any instant makes the angle $(\omega t+P_0OP_1)$ $=(\omega t+\theta_0)$ with the line of reference. Hence the resultant *e. m. f.* is

$$e_1+e_2=E_0 \sin(\omega t+\theta_0)$$

where

$$E_0 = \sqrt{E^2_1 + E^2_2 + 2E_1E_2 \cos \theta} \tag{19a}$$

$$\theta_0 = \tan^{-1}\left[\frac{E_2 \sin \theta}{E_1 + E_2 \cos \theta}\right] \tag{19b}$$

which are identical with equations (18).

The above discussion of course applies to any two *harmonic* functions of the same variable. For example, the resultant of two harmonic currents

$$i_1 = I_1 \sin \omega t$$

and

$$i_2 = I_2 \sin (\omega t + \theta)$$

is

$$i_1 + i_2 = I_0 \sin (\omega t + \theta_0)$$

where

$$I_0 = \sqrt{I_1^2 + I_2^2 + 2I_1I_2 \cos \theta} \tag{20a}$$

$$\theta_0 = \tan^{-1}\left[\frac{I_2 \sin \theta}{I_1 + I_2 \cos \theta}\right] \tag{20b}$$

A re-reading of Articles 8 and 9 will be found helpful in understanding the vector method of representing alternating currents and potential differences.

187. The Vectors Representing Any Number of Harmonic Currents and P.D.'s of the Same Frequency are Stationary with Respect to One Another. — This is immediately evident from the fact that the analytical expression for any harmonic function of the time is $A \sin (\omega t + \theta)$, where A and θ are constants for each such function, and ω, which is equal to 2π times the frequency, is the angular velocity at which the vector representing this function rotates. Hence the vectors representing any number of such functions of the same frequency all rotate through equal angles in any interval of time, *i.e.*, their relative positions with respect to each other remain unchanged. Consequently in problems which involve only *harmonic* currents and *harmonic p.d.'s* and their phase displacements with respect to each other, the vectors representing the currents and *p.d.'s* may be considered as stationary.

It should be carefully noted that *non-harmonic* currents or *p.d.'s*, or currents or *p.d.'s* of *different* frequencies *cannot* be represented on the same diagram by *stationary* vectors.

188. The Lengths of the Vectors Representing Harmonic Functions Taken Equal to their Effective Values. — In any problem which has to do only with the effective values of harmonic

currents, *p.d.'s* and *e.m.f.'s* and their phase relations, the lengths of the vectors representing these quantities may be taken equal to their effective values, since the effective values are directly proportional to the maximum values of these quantities. We then have that

1. The effective value of the resultant of any number of *harmonic* currents (or *p.d.'s* or *e.m.f.'s*) of the same frequency is equal to the *vector* sum of the vectors representing these currents (or *p.d.'s* or *e.m.f.'s*.)

2. The average power corresponding to any *harmonic* current and *p.d.* of the same frequency is equal to the product of the lengths of the vectors representing them times the cosine of the angle between these vectors. See equation (8).

189. Potential Drop due to a Harmonic Current in a Circuit of Constant Resistance and Inductance. — Let r be the resistance

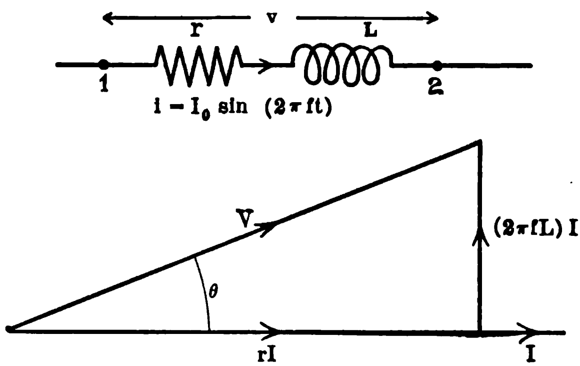


Fig. 101.

and L the inductance between the two points 1 and 2 of the circuit (see Fig. 101), and let this circuit be perfectly insulated and have neither capacity nor mutual inductance with respect to any other circuit. Let the current be $i = I_0 \sin (2\pi ft)$, where I_0 is the maximum value of the current ($=\sqrt{2} \times$ the effective value) and f is the frequency; let v be the value of the instantaneous potential drop from 1 to 2 *in the direction of the current i.* The instantaneous drop of potential in the direction of the current is equal to the resistance drop in the circuit plus the back *e. m. f.* in the coil; compare with equation (18*d*) of chapter III. The resistance drop

is ri and the back *e. m. f.* (due to self inductance) is $L\frac{di}{dt}$ (see Article 116). Hence the total instantaneous potential drop is

$$v = ri + L\frac{di}{dt}$$

Substituting for i its value $I_0 \sin (2\pi f t)$ in the above equation we get

$$v = rI_0 \sin (2\pi f t) + (2\pi f L)\, I_0 \cos (2\pi f t)$$
$$= rI_0 \sin (2\pi f t) + (2\pi f L)\, I_0 \sin \left(2\pi f t + \frac{\pi}{2}\right)$$

Hence when the resistance and inductance are constant the *p.d.* from 1 to 2 is also a harmonic function of the time, has the same frequency, and consists of two components having respectively the effective values rI and $(2\pi f L)\, I$, where $I = \frac{I_0}{\sqrt{2}}$ is the effective value of the current.

The first component of the *p.d.*, namely rI, is in phase with the current and the second component of the *p.d.*, namely $(2\pi f L)I$, leads the current by 90°. Hence the effective value of the resultant *p.d.* is (see the vector diagram and also equations 19)

$$V = \sqrt{(rI)^2 + (2\pi f L I)^2}$$

or

$$V = I\sqrt{r^2 + (2\pi f L)^2} \qquad (21)$$

and this resultant *p.d.* leads the current by the angle

$$\theta = \tan^{-1}\left(\frac{2\pi f L}{r}\right) \qquad (21a)$$

This angle θ is the power-factor angle of the circuit. The power factor of the circuit is then

$$\cos\theta = \frac{r}{\sqrt{r^2 + (2\pi f L)^2}} \qquad (21b)$$

Note that the potential drop in a given portion of a circuit is equal to the *e. m. f. impressed* across the terminals of this portion of the circuit, which *e. m. f.* is in the same direction *around* the closed circuit as the potential drop through the given portion of the circuit; hence in the above formulas V may be taken to represent either the potential drop in the circuit or the *e. m. f. impressed* on it.

Since the first component of the *p.d.* is in phase with the current, and the second component of the *p.d.* is in quadrature with the current, it is the first component alone which determines the average power put into the circuit; rI is therefore the power component of the *p.d.* The electric energy ri^2dt which is converted into heat energy in the resistance r during each interval of time dt is not returned when the current reverses, since this energy is proportional to the *square* of the current i and is therefore independent of the direction of i. The average rate at which this energy is supplied to the circuit is represented by the constant term in the expression for instantaneous power, equation (8).

As just noted, the second component of v, namely $(2\pi fL)I$, is in quadrature with the current and therefore contributes nothing to the *average* power, *i.e.*, this is the reactive component of the *p.d.* This is also evident from the fact that the energy stored in the magnetic field while the current is rising from zero to a maximum is equal to the work done on the current by the field when the current decreases from its maximum value to zero. Also note that this cyclic transfer of energy to and from the magnetic field occurs twice during each cycle of the current, *i.e.*, it is represented by the double frequency term in the expression for instantaneous power, equation (8).

190. Effective Resistance, Reactance and Impedance of an Alternating Current Circuit. — In general when an alternating current is established in an electric circuit, secondary currents are established in the surrounding conductors (for example, as "eddy" currents in the iron cores of the magnetic circuits of electric machinery); a certain amount of energy in addition to that dissipated in the main or primary circuit is therefore also dissipated in the conductors in which these secondary currents are established. Also, due to the fact that the number of lines of magnetic induction linking the center of a wire is greater than the number linking the outside of the wire (see Article 121) the induced back electromotive f inside the wire is in general greater than that induced in the outside filaments of the wire, and therefore the current density is not uniform over the cross section of the wire, as is the case with a continuous current; hence the effective resistance of a conductor to an alternating current is greater than its resistance to a continuous current. This skin effect, however, is not as a rule serious for the frequencies and sizes of conductors used in

practice, except in the case of steel rails used for conductors in railway work. Also, when there is iron in the magnetic field established by the current, a certain amount of energy is dissipated in the iron, due to hysteresis. Hence the average rate at which heat energy is dissipated when an alternating current is established in a given portion of a circuit is not equal to the product of the square of the effective value of the current by the resistance of the conductor in which the current is established, as determined by a continuous current measurement, but is in general greater than this. This portion of the circuit may, however, be considered as having an *effective* or *apparent* resistance r *such that this resistance multiplied by the square of the effective value of the current gives the true average rate at which heat energy is dissipated when the current is established in this portion of the circuit.* This effective resistance is in most practical cases approximately independent of the current strength (as measured by the effective value) but does depend upon the frequency of the current, and, in case there is a loss of energy due to hysteresis, upon the effective value and the wave shape of the current also. (The part of the effective resistance which takes into account the hysteresis loss in iron is not strictly constant, but varies approximately as the 1.6 power of the maximum flux density, and therefore as the 1.6 power of the *maximum* value of the current.)

The average rate at which heat energy is dissipated when an alternating current is established in any portion of a circuit may then be written

$$P_h = rI^2 \qquad (22)$$

where r is the effective resistance of this portion of the circuit and I is the effective value of the current.

The effective value of the resultant potential drop in any portion of a circuit, however, is in general greater than rI. Let V be the vector representing the resultant *p.d.* in any portion of the circuit and let E be the vector representing any externally induced *e. m. f.* in this portion of the circuit, both in the direction of the current (*e.g.*, if the portion of the circuit considered is the armature of an alternator, E is the induced *e. m. f.* due to the relative motion of the armature and the magnetic field). Then the ratio of the numerical value of the vector difference $\overline{E-V}$ to the effective value of the current I in this portion of the circuit is

defined as the *impedance* of this portion of the circuit. That is, the impedance of this portion of the circuit is

$$z=\frac{\overline{E-V}}{I} \tag{23}$$

When there is no externally induced *e. m. f.*, that is, when E is zero, the impedance of the given portion of the circuit is

$$z=\frac{V}{I} \tag{23a}$$

The square root of the difference between the square of the impedance of any portion of a circuit and the square of the effective resistance of this portion of the circuit is called the *reactance* of this portion of the circuit, and is usually represented by the symbol x. The reactance corresponding to the impedance z and the effective resistance r is then

$$x=\sqrt{z^2-r^2} \tag{24}$$

In the case of a harmonic current in a given portion of circuit in which there is no externally induced electromotive force, rI is the component of the *p.d.* in phase with the current, and xI is the component of the *p.d.* in quadrature with the current. When the current *lags behind* the potential drop in the direction of the current the reactance is taken as *positive*, when the current *leads* the *p.d.* the reactance is taken as *negative;* that is, the sign of x is chosen so that xI represents the component of the *p.d.* 90° *ahead* of the current. The angle by which the current lags behind the *p.d.*, or by which the *p.d.* leads the current, is then

$$\theta=tan^{-1}\frac{x}{r} \tag{25}$$

and the power factor of the circuit is

$$\cos\theta=\frac{r}{\sqrt{r^2+x^2}}=\frac{r}{z} \tag{25a}$$

It also follows from the definition of reactive power (Article 184), that the reactive power is equal to xI^2.

Since both reactance and impedance are ratios of *p.d.* to current they are both measured in the same unit as resistance, *i.e.*, in practical units both the reactance and the impedance are expressed in ohms.

191. Reactance and Impedance of a Coil of Constant Resistance and Inductance* to a Harmonic Current. — From equations (21) and the above definitions, it is evident that the impedance of a circuit of constant resistance (independent of the value of the current at any instant) and a constant inductance (independent ot the value of the current at each instant), to a harmonic current is

$$z=\sqrt{r^2+(2\,\pi\,fL)^2} \tag{26}$$

and the reactance is

$$x=2\,\pi\,fL \tag{26a}$$

Note that these relations are deduced on the assumptions that the coil has no electrostatic capacity and that the current is a sine wave.

192. Reactance and Impedance of a Coil of Constant Resistance and Inductance to a Non-Harmonic Current. — The reactance of such a circuit to a non-harmonic current is readily found when the equation of the current is known. Consider the special case of a current which contains the third harmonic. This current may be represented by the equation

$$i=I_1\,sin\,(2\,\pi\,ft)+I_3\,sin\,(6\,\pi\,ft+\theta)$$

where I_1 and I_3 are the maximum values of the fundamental and third harmonic respectively. When the wave shape remains constant for all values of the current $\frac{I_3}{I_1}$ is a constant. Then the instantaneous value of the drop of potential through the resistance and inductance is

$$\begin{aligned} v=ri+L\frac{di}{dt}&=rI_1\,sin\,(2\,\pi\,ft)+rI_3\,sin\,(6\,\pi\,ft+\theta)\\ &+(2\,\pi\,fL)\,I_1\,cos\,(2\,\pi\,ft)+(6\,\pi\,fL)\,I_3\,cos\,(6\,\pi\,ft+\theta)\end{aligned}$$

Combining the terms of the same frequency in the manner described in Article 185, we get

$$\begin{aligned} v&=\sqrt{r^2+(2\,\pi\,fL)^2}\,I_1\,sin\,(2\,\pi\,ft+a_1)\\ &+\sqrt{r^2+(6\,\pi\,fL)^2}\,I_3\,sin\,(6\,\pi\,ft+a_2)\end{aligned}$$

where

* Such a coil is frequently called an "impedance" coil.

$$a_1 = tan^{-1}\frac{2\pi fL}{r} \text{ and } a_2 = \theta + tan^{-1}\frac{6\pi fL}{r}$$

(see equation 18*b*).

The effective value of V is then (see Article 176)

$$V = \sqrt{\left[r^2 + (2\pi fL)^2\right]\frac{I_1^2}{2} + \left[r^2 + (6\pi fL)^2\right]\frac{I_3^2}{2}}$$

and the effective value of i is

$$I = \sqrt{\frac{I_1^2}{2} + \frac{I_3^2}{2}}$$

whence

$$\frac{V}{I} = \sqrt{r^2 + (2\pi fL)^2\left[1 + 8\frac{I_3^2}{I_1^2 + I_3^2}\right]}$$

Since $\frac{I_3}{I_1}$ is constant, $\frac{I_3^2}{I_1^2 + I_3^2}$ is constant; put $a^2 = \frac{I_3^2}{I_1^2 + I_3^2}$. Then the impedance of the circuit to this current is equal to

$$z = \sqrt{r^2 + (2\pi fL)^2 (1 + 8a^2)} \qquad (27)$$

and the reactance is

$$x = \sqrt{z^2 - r^2} = 2 \cdot \pi fL\sqrt{1 + 8a^2} \qquad (27a)$$

The constant a is the ratio of the effective value of the third harmonic to the effective value of the resultant current.

Note that the effective value of the first harmonic in the *p.d.* wave is $\frac{I_1}{\sqrt{2}}\sqrt{r^2 + (2\pi fL)^2}$, while the effective value of the third harmonic is $\frac{I_3}{\sqrt{2}}\sqrt{r^2 + (6\pi fL)^2}$. When the inductance L is large compared with the resistance, the ratio of the third harmonic in the *p.d.* wave to its fundamental must then be three times as great as the ratio of the third harmonic to the fundamental in the current wave. *Vice versa,* the third harmonic in the current wave resulting from a given *p.d.* will be relatively only one third as great in the current wave as in the *p.d.* wave. Therefore an inductance in a circuit tends to dampen out the harmonics in the current wave when a non-harmonic *e. m. f.* is impressed on it, and to make this wave approach more nearly to a sine wave. The higher the order of the harmonic the greater the damping.

193. Current through a Condenser when a Harmonic P.D. is Established across It. — Capacity Reactance. — Let the *p.d.* across the terminals of the condenser be

$$v = V_0 \sin 2 \pi ft$$

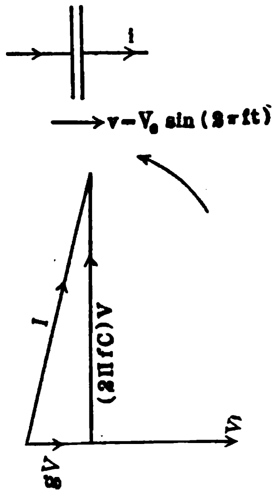


Fig. 102.

and let the total current through the condenser in the direction of the potential drop be i. Let C be the capacity of the condenser and g the conductance of the dielectric between its plates, *i.e.*, the leakance of the condenser. Then, since the total current through the condenser is the sum of the conduction current, gv, through the dielectric plus the displacement current $C\frac{dv}{dt}$ through the dielectric (see Article 151), we have

$$i = gv + C\frac{dv}{dt}$$

Therefore

$$i = g V_0 \sin (2 \pi ft) + (2 \pi fC) V_0 \cos (2 \pi ft)$$
$$= g V_0 \sin (2 \pi ft) + (2 \pi fC) V_0 \sin \left(2 \pi ft + \frac{\pi}{2}\right)$$

Hence the component of the current through the condenser due to the conductance of the dielectric, *i.e.*, the leakage current, is in phase with the *p.d.* across the condenser, and the component of the current through the condenser due to its capacity, *i.e.*, the displacement or " charging " current, *leads* the *p.d.* by 90°. The effective value I of the *total* current is

$$I = V\sqrt{g^2 + (2 \pi fC)^2}$$

where V is the effective value of the *p.d.* Hence the effective resistance of a leaky condenser to an alternating current of frequency f is (see Article 190)

$$r_c = \frac{gV^2}{I^2} = \frac{g}{g^2 + (2 \pi fC)^2} \qquad (28)$$

and its impedance is

$$z_c = \frac{V}{I} = \frac{1}{\sqrt{g^2 + (2 \pi fC)^2}} \qquad (28a)$$

and therefore its reactance is

$$x_c = \sqrt{z^2 - r^2} = -\frac{2\,\pi\, fC}{g^2 + (2\,\pi\, fC)^2} \qquad (28b)$$

x_c is negative, since the current leads the *p.d.* When the dielectric is a perfect insulator $g=0$, and in this case $r_c=0$ and the reactance of the condenser reduces to

$$x_c = -\frac{1}{2\,\pi\, fC} \qquad (28d)$$

The reactance $-\dfrac{I}{2\,\pi\, fC}$ corresponding to a capacity alone, without leakance, is called the *capacity reactance* of the condenser; it must not be confused with the *effective* reactance of the condenser; the two are equal only when there is no leakage. The leakage conductance of condensers used in practice is usually quite small but is not always negligible.

194. Current through a Condenser when a Non-Harmonic P.D. is Established across It. — When the *p.d.* contains the third harmonic, for example, its equation is

$$v = V_1 \sin (2\,\pi\, ft) + V_3 \sin (6\,\pi\, ft + \theta).$$

Neglecting the leakance of the condenser, the current is then

$$i = C\frac{dv}{dt}$$

$$= (2\,\pi\, fC)\; V_1 \cos 2\,\pi\, ft + (6\,\pi\, fC)\, V_3 \cos (6\,\pi\, ft + \theta)$$

The effective value of the *p.d.* is, see Article 176,

$$V = \sqrt{\frac{V_1^2}{2} + \frac{V_3^2}{2}}$$

and the effective value of the current is, see Article 176,

$$I = (2\,\pi\, fC)\sqrt{\frac{V_1^2}{2} + 9\,\frac{V_3^2}{2}} = (2\,\pi\, fC)\, V\sqrt{1+8a^2} \qquad (29)$$

where $a = \sqrt{\dfrac{V_3^2}{V_1^2 + V_3^2}}$ = ratio of the effective value of the third harmonic of the *p.d.* to the resultant *p.d.*

When the leakance of the condenser is negligible the ratio of the third harmonic to the fundamental in the current wave is therefore three times as great as the ratio of the third harmonic to the fundamental in the *p.d.* wave. Hence, when a non-harmonic *p.d.* is impressed across a condenser, the upper harmonics in the current wave are greater than the corresponding

harmonics in the *p.d.* wave directly in proportion to their order. For example, when the seventh harmonic in the *p.d.* wave has an amplitude equal to 5 per cent of the fundamental, the seventh harmonic in the current wave has an amplitude equal to 35 per cent of the amplitude of the fundamental in the current wave. Compare with the effect produced by an inductance, Article 192.

195. Impedance of a Resistance, Inductance and Capacity in Series to a Harmonic Current. — When a harmonic current of effective value I is established in such a circuit (see Fig. 103), the *p.d.* across the resistance is $V_r = rI$ and is in phase with I. The *p.d.* across the inductance is $V_L = (2\pi fL)\,I$ and *leads* I by 90°. The *p.d.* across the capacity (a condenser with *negligible* leakance) is $V_c = \dfrac{I}{2\pi fC}$ and *lags behind* I by 90°. Hence the resultant *p.d.* is

$$V = \overline{V_r + V_L + V_C} = I\sqrt{r^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2} \qquad (30)$$

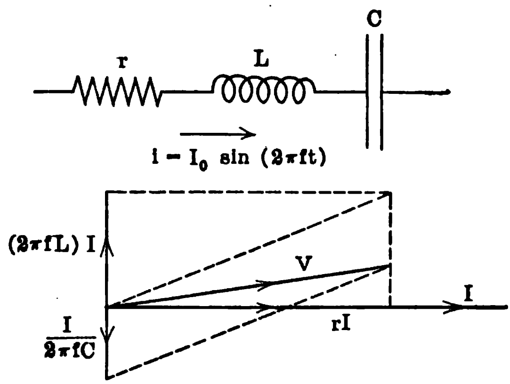


Fig. 103.

The impedance of such a circuit is therefore

$$z = \frac{V}{I} = \sqrt{r^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2} \qquad (30a)$$

The reactance is therefore

$$x = \sqrt{z^2 - r^2} = \left(2\pi fL - \frac{1}{2\pi fC}\right) \qquad (30b)$$

The angle by which the *p.d.* leads the current is

$$\theta = \tan^{-1}\frac{x}{r} \qquad (30c)$$

Note that equations (30) apply only in case the condenser has no leakance.

196. Resonance. — When the inductance L and capacity C in the case just considered are of such values that

$$2\pi fL=\frac{1}{2\pi fC}$$

or when

$$f=\frac{1}{2\pi\sqrt{LC}} \tag{31}$$

the reactance of the circuit is zero and the impedance is equal to the resistance, and therefore the current corresponding to a given *p.d.* V is $I=\frac{V}{r}$; that is, the current is a maximum and depends only upon the resistance of the circuit. The frequency corresponding to this condition is the same as the frequency with which the current and *p.d.* would oscillate were the condenser short-circuited by the inductance; *i.e.*, this frequency corresponds to the free period of such a circuit. (See Article 164.) Note the analogy with the motion of a body, which is free to vibrate, produced by a periodic force having a period equal to the period of vibration of the body; for example, a heavy church bell may be caused to swing with large amplitude (corresponding to the maximum value of the current I) when a comparatively small force (corresponding to the *p.d.* V) is applied by the man pulling the rope, provided the successive applications of this force are in time with the swinging of the bell.

When the frequency of the electromotive force impressed across the terminals of the circuit is equal to the natural or free frequency of the circuit, the circuit is said to be *in resonance* with this impressed electromotive force.

Note also that, although the resultant *p.d.* across the resistance, inductance and capacity in series, is equal to rI, *i.e.*, is the same as the *p.d.* across the resistance, the *p.d.* across the inductance or across the condenser may be many times this. For example, when the inductance L is 1 henry and the capacity C is 7.04 microfarads, and the frequency is 60 cycles, the inductive reactance is $x_L=2\pi\times 60\times 1=377$, and the capacity reactance is $x_c=-\frac{1}{2\pi\times 60\times 7.04\times 10^{-6}}=-377$. Hence the total reactance

of the circuit is $x_L + x_c = 0$, and the circuit is in resonance with the impressed *e. m. f.* When the resistance r is 1 ohm and the impressed *e. m. f.* across the entire circuit is 100 volts, the current is

$$I = \frac{100}{\sqrt{(1)^2 + (0)^2}} = 100 \text{ amperes}$$

The *p.d.* across the resistance is then $100 \times 1 = 100$ volts. The *p.d.* across the inductance, however, is $2\,\pi \times 60 \times 1 \times 100 = 37{,}700$ volts and the *p.d.* across the condenser is

$$\frac{100}{2\,\pi \times 60 \times 7.04 \times 10^{-6}} = 37{,}700 \text{ volts}$$

This brings out in a striking manner the fundamental fact that *alternating p.d.'s cannot be added algebraically; they must be added vectorially.*

197. Impedances in Series. — In the case of continuous currents we have seen that several resistances r_1, r_2, r_3, etc., in series are equivalent to a single resistance R which is equal to the arithmetical sum of r_1, r_2, r_3, etc., that is $R = r_1 + r_2 + r_3 +$ ——. In case of alternating currents this same relation also holds for the *resistance* of any number of impedances in series, such as a number of coils of wire in series. For, when the same current I flows through each coil the average power dissipated in all the coils is $r_1I^2 + r_2I^2 + r_3I^2 + \text{etc.} = (r_1 + r_2 + r_3 + \text{etc.})I^2$ where r_1, r_2, r_3, etc., are the effective resistances of the respective coils. Hence, calling R the effective resistance of all the coils in series, we have from Article 190 that

$$R = r_1 + r_2 + r_3 + \text{etc.} \tag{32a}$$

In the case of a harmonic current the total potential drop through all the coils *in phase* with the current is then $RI = (r_1 + r_2 + r_3 + \text{etc.})I$.

Similarly, calling x_1 x_2, x_3, etc., the reactances of the respective coils to the harmonic current I, the *p.d.'s* across the separate reactances are respectively x_1I, x_2I, x_3I, etc., and are all in *quadrature with the current* and are therefore either in the same or opposite direction. Hence the algebraic sum of these *p.d.'s* gives the total *p.d.* in quadrature *ahead* of the current. Whence, calling X the equivalent reactance of the circuit, we have from Article 190 that

$$XI = x_1I + x_2I + x_3I + \text{etc.}$$

or

$$X = x_1 + x_2 + x_3 + \text{etc.} \tag{32b}$$

(Note that the reactances x may be either positive or negative; an inductive reactance is positive, a capacity reactance negative.)

The resultant *p.d.* across all the coils in series is therefore

$$V=\sqrt{(RI)^2+(XI)^2}$$
$$=\sqrt{(r_1I+r_2I+r_3I+\text{etc.})^2+(x_1I+x_2I+x_3I+\text{etc.})^2}$$

or

$$V=I\sqrt{R^2+X^2}=I\sqrt{(r_1+r_2+r_3+\text{etc.})^2+(x_1+x_2+x_3+\text{etc.})^2}$$

Whence, the equivalent impedance of such a circuit is, from Article 190,

$$Z=\frac{V}{I}=\sqrt{R^2+X^2}=\sqrt{(r_1+r_2+r_3+\text{etc.})^2+(x_1+x_2+x_3\ \text{etc.})^2} \quad (32c)$$

Hence, the equivalent impedance of any number of impedances z_1, z_2, z_3, etc., is *not* the sum of the separate impedances. In general, *the equivalent impedance can be calculated only when the resistance r and the reactance x of each impedance is known.*

Example: An alternating current of 100 amperes is to be supplied to a receiver which has an equivalent resistance r_1 of 2 ohms and an equivalent reactance x_1 of 0.5 ohm. The line has a resistance r_2 of 0.1 ohm and an inductive reactance x_2 of 1.5 ohms. The equivalent resistance of the line and receiver is then $R=2+0.1=2.1$ ohms and the equivalent reactance of the line and receiver is $X=0.5+1.5=2.0$ ohms. Hence the equivalent impedance of the line and receiver is $Z=\sqrt{(2.1)^2+(2.0)^2}=2.90$ ohms. The impedance of the receiver alone is $z_1=\sqrt{(2)^2+(0.5)^2}$ $=2.06$ and the impedance of the line alone is $z_2=\sqrt{(0.1)^2+(1.5)^2}$ $=1.50$. Hence $z_1+z_2=3.56$ which is 23 per cent greater than the true impedance of the line and receiver.

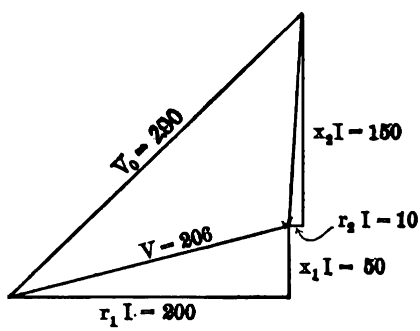


Fig. 104.

When the current supplied to the receiver is 100 amperes, the *p.d.* at the receiver is $V=100\times z_1=100\times 2.06=206$ volts and the *p.d.* at the generator is $V_0=100\times Z=100\times 2.90=290$ volts; that is, the *p.d.* at the receiver is $290-206=84$ volts less than at the generator. The total potential drop in the two wires forming the line, however, is 100

$\times z_2 = 100 \times 1.50 = 150$ volts, which is 79 per cent greater than the true difference between the potential drops across the generator and across the receiver terminals.

Fig. 104 will make these relations clear. The reason the *p.d.* in the line is not equal to the difference of the *p.d.'s* at the generator and the receiver is that the *p.d.* in the line and the *p.d.* at the receiver are *not in phase.*

198. Impedances in Parallel. — In the case of continuous currents we have seen (Article 98) that when several resistances r_1, r_2, r_3, etc., are connected in parallel, the currents in the various resistances are respectively $I_1 = \frac{V}{r_1}$, $I_2 = \frac{V}{r_2}$, $I_3 = \frac{V}{r_3}$, etc., and therefore that the total current between the junction points of the several resistances is $I = I_1 + I_2 + I_3 +$ etc., $= V\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + - -\right)$.

Whence the equivalent resistance R must be such that

$$\frac{1}{R} = \frac{I}{V} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + - - -$$

In the case of alternating currents, the currents in any number

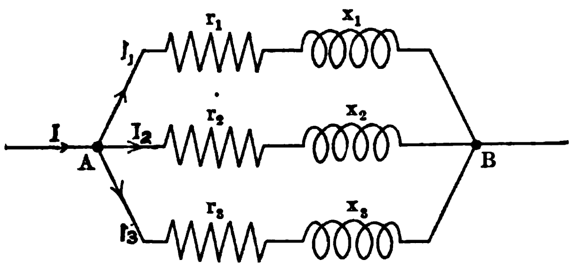


Fig. 105

of impedances connected in parallel between two points A and B (Fig. 105) are respectively

$$I_1 = \frac{V}{z_1}$$

$$I_2 = \frac{V}{z_2}$$

$$I_3 = \frac{V}{z_3} \text{ etc.}$$

but these currents are *not in phase*, hence they cannot be added

algebraically; they must be added vectorially. The components of the various currents in phase with the potential drop V between A and B are respectively

$$I_1 \cos\theta_1 = \frac{V}{z_1}\frac{r_1}{z_1} = \frac{r_1}{z_1^2}V$$

$$I_2 \cos\theta_2 = \frac{V}{z_2}\frac{r_2}{z_2} = \frac{r_2}{z_2^2}V \text{ etc.}$$

and the components of these currents in quadrature *behind* V are respectively

$$I_1 \sin\theta_1 = \frac{V}{z_1}\frac{x_1}{z_1} = \frac{x_1}{z_1^2}V$$

$$I_2 \sin\theta_2 = \frac{V}{z^2}\frac{x_2}{z_2} = \frac{x_2}{z_2^2}V \text{ etc.}$$

The total current *in phase* with V is then

$$\left[\frac{r_1}{z_1^2}+\frac{r_2}{z_2^2}+---\right]V$$

and the total current *in quadrature behind* V is

$$\left[\frac{x_1}{z_1^2}+\frac{x_2}{z_2^2}+---\right]V$$

Hence the total current I is the vector sum of these two components, *i.e.*,

$$I = V\sqrt{\left[\frac{r_1}{z_1^2}+\frac{r_2}{z_2^2}+---\right]^2+\left[\frac{x_1}{z_1^2}+\frac{x_2}{z_2^2}+---\right]^2}$$

and therefore the equivalent impedance is Z, where

$$\frac{1}{Z}=\frac{I}{V}=\sqrt{\left[\frac{r_1}{z_1^2}+\frac{r_2}{z_2^2}+--\right]^2+\left[\frac{x_1}{z_1^2}+\frac{x_2}{z_2^2}+--\right]^2} \tag{33}$$

199. Conductance, Susceptance and Admittance. — Note that for any of the impedances the factor $\frac{r}{z^2}$ multiplied by the *p.d.* across the impedance gives the component of the current in phase with the *p.d.* and therefore this factor $\frac{r}{z^2}$ multiplied by the square of the *p.d.*, *i.e.*, $\frac{rV^2}{z^2}$, gives the average power dissipated in this impedance. In general, the ratio of the average rate P_h at which heat energy is dissipated in any part of a circuit to the square of the effective value of the *p.d.* V across this part of the circuit, is

called the effective *conductance* "g" of this part of the circuit, that is,

$$g = \frac{P_h}{V^2} \tag{34}$$

The ratio of the effective value of the current in any part of a circuit to the numerical value of the vector difference of the externally induced *e. m. f.* E and the resultant *p.d.* V in this portion of the circuit is called the *admittance* "y" of this part of the circuit, that is,

$$y = \frac{I}{E - V} \tag{35}$$

The square root of the difference between the square of the admittance of any part of a circuit and the square of the effective conductance of this part of the circuit is called the *susceptance* "b" of this part of the circuit, that is,

$$b = \sqrt{y^2 - g^2} \tag{36}$$

In the case of a harmonic *p.d.*, established across the terminals of a circuit in which there is no externally induced *e. m. f.*, the product of the susceptance and the *p.d.* gives the component of the current in quadrature with the *p.d.*, since the total current is yV and the component in phase with V is gV, whence the component in quadrature with V is $\sqrt{(yV)^2 - (gV)^2} = V\sqrt{y^2 - g^2}$. The sign of the susceptance b is taken positive when the *p.d.* leads the current, negative when the *p.d.* lags behind the current; that is, the susceptance and reactance of a circuit always have the same sign. Also, from the definition of reactive power (Article 184), it follows that the reactive power is bV^2.

From the above definitions and the discussion in Article 193, it follows that the effective conductance, or as it is also called, the leakance, of a condenser, is equal to the reciprocal of the resistance of the dielectric between its plates, as measured by means of a continuous current (provided there is no dissipation of energy due to "dielectric hysteresis") and the susceptance of a condenser of capacity C to a harmonic current of frequency f is

$$b_c = -2\pi fC.$$

The susceptance of a condenser is negative, since the charging current corresponding to the capacity C leads the *p.d.* The admittance of a condenser having a leakance g and capacity C to a harmonic current of frequency f is then

$$y_c=\sqrt{g^2+(2\ \pi\ fC)^2}$$

Compare with the resistance, reactance and impedance of a coil.

Also, from the above definitions, we have in general that, corresponding to an impedance $z=\sqrt{r^2+x^2}$, the conductance is

$$g=\frac{r}{z^2} \qquad (37a)$$

the admittance is

$$y=\frac{1}{z}=\sqrt{g^2+b^2} \qquad (37b)$$

and the susceptance is

$$b=\frac{x}{z^2} \qquad (37c)$$

Hence the resultant admittance of any number of circuits in parallel, equation (33), is

$$Y=\sqrt{(g_1+g_2+g_3--)^2+(b_1+b_2+b_3--)^2} \qquad (37d)$$

where g_1, g_2, g_3, etc., and b_1, b_2, b_3, etc., are the conductances and susceptances respectively of the various circuits.

Since conductance, susceptance and admittance are the ratios of current to voltage, the unit in which these quantities are measured is the reciprocal of the ohm; these quantities are therefore expressed in "mhos."

For circuits in series, it is more convenient to use the quantities resistance, reactance and impedance. For circuits in parallel, the conductance, susceptance and admittance are more convenient, since conductances and susceptances of parallel circuits are respectively additive. Note that when g and b are given r and x can be immediately calculated, since

$$r=\frac{g}{y^2} \qquad (38)$$

and

$$x=\frac{b}{y^2} \qquad (38a)$$

Compare with equations (37).

200. Admittance of an Inductance and Capacity in Parallel to a Harmonic Current.— Resonance.— Let L be the inductance of a coil of negligible resistance, C the capacity of a condenser having

negligible leakance, and f the frequency of the current. Let the coil and condenser be connected in parallel as shown in Fig. 106.

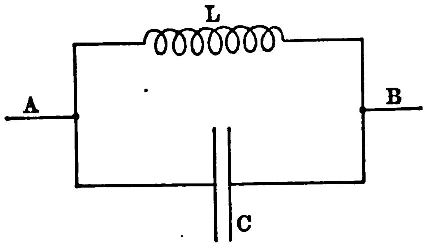


Fig. 106.

Then, the susceptance of the coil is $b = \dfrac{1}{2\pi f L}$; and the susceptance of the condenser is $b_c = -2\pi f C$. Hence the admittance of the inductance and capacity in parallel is

$$Y = \sqrt{(0+0)^2 + \left(\frac{1}{2\pi f L} - 2\pi f C\right)^2}$$

$$= \frac{1}{2\pi f L} - 2\pi f C$$

Hence the total current from A to B when a drop of potential V is established from A to B is

$$I = YV = V\left(\frac{1}{2\pi f L} - 2\pi f C\right)$$

When $\dfrac{1}{2\pi f L} = 2\pi f C$, that is, when $f = \dfrac{1}{2\pi\sqrt{LC}}$ the total admittance is zero and therefore the total current from A to B is zero. This is also evident from the fact that the current in the inductance is $\dfrac{V}{2\pi f L}$ and *lags behind* the *p.d.* by 90°; the current in the condenser is $2\pi f CV$ and *leads* the *p.d.* by 90°. Hence, when $2\pi f C = \dfrac{1}{2\pi f C}$, the current in the inductance at each instant is exactly equal and opposite to the current in the condenser, and therefore the total current is zero. Note that the frequency $f = \dfrac{1}{2\pi\sqrt{LC}}$ is the natural frequency of the closed circuit formed by the condenser and the inductance. (Compare with Article 196.)

The above deductions are based on the assumption that a *harmonic* or *sine-wave* current is established in both the capacity and in the inductance. In the ideal case of a coil of no resistance, a harmonic *p.d.* impressed across its terminals does not necessarily

establish a harmonic current, but the expression for the current may contain a constant term. In the case of any actual coil, however, the resistance of the coil (which can never be actually zero though it may be quite small) causes the constant term in the expression for the current to become zero after a short time, and the current becomes a true harmonic current, see Article 201. Also, due to the resistance of the coil, and the leakance of the condenser as well, the resultant admittance of a coil and condenser in parallel can never be zero and hence for a given *p.d.* across their common terminals the total current can never be absolutely zero. However, when the resistance of the coil and the leakance of the condenser are small compared with the reactance of the coil and the susceptance of the condenser, the total current will be a minimum when the impressed *e. m. f.* has the frequency

$$f=\frac{1}{2\pi\sqrt{LC}}$$

Example: When $L=1$ henry and $C=7.04$ microfarads, and the frequency f is 60 cycles per second, $\frac{1}{2\pi fL}=\frac{1}{377}$ and $2\pi fC=\frac{1}{377}$ and therefore the total admittance, neglecting the resistance and leakance, is zero. Hence, when a coil of inductance L and a condenser of capacity C, connected in parallel, are connected to a 60-cycle generator and a sufficient time is allowed for the current in both the condenser and the coil to become true harmonic currents, a potential drop of 100 volts across the common terminals A and B will establish a current of $\frac{100}{377}=0.265$ amperes in the condenser and also in the inductance, provided the resistance of the coil and the leakance of the condenser are small compared with the reactance of the coil and the susceptance of the condenser respectively.

201. Transient Effects Produced when a Harmonic E. M. F. is Impressed on a Circuit. — So far, we have considered the relation between current and *p.d.* in various types of circuits when a harmonic alternating current is established in the circuit. However, when a harmonic *e. m. f.* is impressed across the terminals of a circuit, time is required for a harmonic current to become established just as in the case of a constant *e. m. f.* impressed upon a

circuit time is required for a continuous current to become established. (See Article 160.) That is, when an alternating *e. m. f.* is impressed across the terminals of a circuit, the current at the start is an *oscillating* current (see Article 167) and becomes an alternating current with fixed maximum positive and negative values only after the lapse of an appreciable, though usually

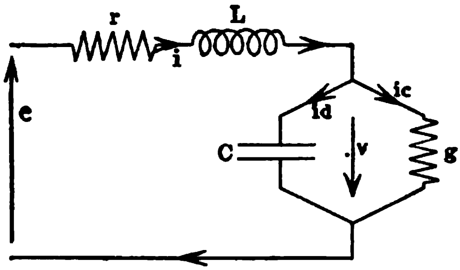


Fig. 107.

small, interval of time. Let the circuit be as shown in Fig. 107; r and L represent the resistance and inductance of an impedance coil, and C and g the capacity and the leakance of a condenser Let the impressed *e. m. f.* be $e=E\ sin\ (2\pi ft+\beta)$. The general differential equations of this circuit have already been given (Article 159) and are

$$i=gv+C\frac{dv}{dt} \qquad (a)$$

$$v=e-ri-L\frac{di}{dt} \qquad (b)$$

where i is the instantaneous current in the impedance coil (equal to the total displacement and leakage current of the condenser) and v is the *p.d.* across the condenser in the direction of the current. Substituting in (a) the value of v from (b) we get

$$LC\frac{d^2 i}{dt^2}+(rC+gL)\frac{di}{dt}+(1+rg)\ i=ge+C\frac{de}{dt} \qquad (c)$$

This is a differential equation of the second order. It is integrated by finding first the complementary function, *i.e.*, the solution corresponding to the right-hand number equal to zero, and then adding to this the particular integral. The solution corresponding to the right-hand number zero is $i=A\epsilon^{at}$, where the value of a is found by substituting this value of i in the equation (c), which gives

$$LCa^2+(rC+gL)\,a+(1+rg)=0$$

whence

$$a=-\frac{1}{2}\left(\frac{r}{L}+\frac{g}{C}\right)\pm\sqrt{\frac{1}{4}\left(\frac{r}{L}-\frac{g}{C}\right)^2-\frac{1}{LC}}$$

Put

$$u=\frac{1}{2}\left(\frac{r}{L}+\frac{g}{C}\right)\text{ and }m=\sqrt{\frac{1}{4}\left(\frac{r}{L}-\frac{g}{C}\right)^2-\frac{1}{LC}} \qquad (d)$$

Then the complementary function is

$$\epsilon^{-ut}\left(A_1\epsilon^{mt}+A_2\epsilon^{-mt}\right) \qquad (e)$$

where A_1,and A_2 are constants of integration.

The particular integral of equation (c) is of the form $i=I_0\,sin(2\,\pi\,ft+\beta-\theta)$, where

$$I_0=\frac{E}{\sqrt{\left[r+\dfrac{g}{g^2+b^2}\right]^2+\left[x+\dfrac{b}{g^2+b^2}\right]^2}} \qquad (f)$$

and

$$\theta=tan^{-1}\left\{\frac{x+\dfrac{b}{g^2+b^2}}{r+\dfrac{g}{g^2+b^2}}\right\} \qquad (g)$$

where $b=-2\pi fC$ and $x=2\,\pi\,fL$. That this is true can be seen by substituting $i=I_0\,sin\,(2\pi ft+\beta-\theta)$ in equation (c), putting for I_0 and θ the values given by (f) and (g).

Note that $x+\dfrac{b}{g^2+b^2}$ represents the equivalent reactance and $r+\dfrac{g}{g^2+b^2}$ represents the equivalent resistance of the entire circuit. Hence the particular integral $I_0\,sin\,(2\,\pi\,ft+\beta-\theta)$ represents a current equal to the impressed *e. m. f.* divided by the equivalent impedance of the entire circuit and lags behind the impressed *e. m. f.* by the angle corresponding to the power-factor angle of the entire circuit. That is, the particular integral is the final alternating current which is established in the circuit by the impressed *e. m. f.*

The complete solution of the current equation (c) is then

$$i=\epsilon^{-ut}\left[A_1\epsilon^{mt}+A_2\epsilon^{-mt}\right]+\frac{E}{\sqrt{R^2+X^2}}\sin\left(2\pi ft+\beta-\tan^{-1}\frac{X}{R}\right) \tag{39}$$

where

$$u=\frac{1}{2}\left(\frac{r}{L}+\frac{g}{C}\right)$$

$$m=\sqrt{\frac{1}{4}\left(\frac{r}{L}-\frac{g}{C}\right)^2-\frac{1}{LC}}$$

$$R=r+\frac{g}{g^2+(2\pi fC)^2}$$

$$X=x-\frac{2\pi fC}{g^2+(2\pi fC)^2}$$

The constants of integration A_1 and A_2 are determined from the values of the current i_0 and the *p.d.* v_0 across the condenser at time $t=0$.

Since the constant u is always greater than the constant m, the transient term $\epsilon^{-ut}[A_1\epsilon^{mt}+A_2\epsilon^{-mt}]$ in equation (39) becomes smaller and smaller as time increases. In most practical cases, only a fraction of a second is necessary for this term to become negligible in comparison with the term $\frac{E}{\sqrt{R^2+X^2}}\sin\left(2\pi ft+\beta-\tan^{-1}\frac{X}{R}\right)$ representing the true alternating current. Hence the transient term is usually neglected, as most problems which arise in ordinary practical work have to deal only with the steady state produced in a circuit when a given *e.m.f.* is impressed upon it. However, there are cases in which the transient term becomes of paramount importance; in particular, in predetermining the effect of switching on or off a heavy load from a transmission line, the short-circuiting or grounding of a line, and the effect of lightning discharges.

In most practical problems the leakance of the condenser is negligible, that is, $g=0$. When this condition holds, the constants in equation (39) have the values

$$u=\frac{r}{2L}$$

$$m=\frac{1}{2L}\sqrt{r^2-\frac{4L}{C}}=\sqrt{u^2-\frac{1}{LC}}$$

$$R=r$$

$$X = x - \frac{1}{2\,\pi f C}$$

202. Discharge of a Condenser having Negligible Leakance through a Resistance and Inductance. — As an example of the application of equation (39), the manner in which a condenser of capacity C (Fig. 108) charged to a *p.d.* V discharges through a circuit containing a resistance r and inductance L will be determined. (Compare with the ideal case of the discharge of a condenser through an inductance having no resistance, Article 164.) At time $t=0$, the switch S is closed. The impressed *e. m. f.* is therefore zero, hence the equation for the current is

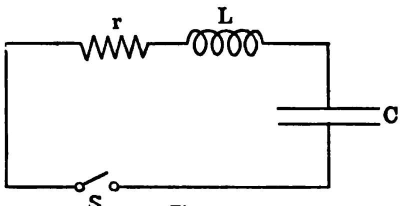


Fig. 108.

$$i = \epsilon^{-ut}\left[A_1\epsilon^{mt} + A_2\epsilon^{-mt}\right] \qquad (a)$$

where A_1 and A_2 are constants to be determined from the initial current and *p.d.* across the condenser.

If there is originally no current in the circuit, the current must start from zero (otherwise there would be an instantaneous transfer of a finite amount of energy, $\frac{1}{2}Li^2$, to the magnetic field set up by the current, which would mean an infinite rate of transfer, or infinite power, which is impossible) and therefore at time $t=0$, $i=0$. We also have from the relation $v + ri + L\frac{di}{dt} = 0$, that at time $t=0$,

$$V = -L\frac{di}{dt} \text{ or } \left[\frac{di}{dt}\right]_{t=0} = -\frac{V}{L}$$

Hence, substituting these conditions in equation (a), we have that

$$0 = A_1 + A_2$$

$$-\frac{V}{L} = (m-u)\,A_1 - (m+u)\,A_2$$

Whence, solving these two equations, we get

$$A_1 = -\frac{V}{2mL} \text{ and } A_2 = +\frac{V}{2mL}$$

These values of A_1 and A_2 substituted in (a) give

$$i = -\frac{V}{2mL}\epsilon^{-ut}\left[\epsilon^{mt} - \epsilon^{-mt}\right] \qquad (40)$$

Since the leakance g of the condenser is assumed to be zero, u and m have the values

$$u = \frac{r}{2L}$$

$$m = \sqrt{u^2 - \frac{1}{LC}}$$

u is therefore always real, but m may be either real or imaginary, depending upon whether $u^2 > \frac{1}{LC}$ or whether $u^2 < \frac{1}{LC}$, that is, upon whether $r^2 > \frac{4L}{C}$ or $r^2 < \frac{4L}{C}$.

For $r^2 > \frac{4L}{C}$, the constant m is real and equation (40) may therefore be written

$$i = -\frac{V}{mL}\epsilon^{-ut}\sinh(mt) \qquad (40a)^*$$

This equation tells us that the current starts at zero, rises to

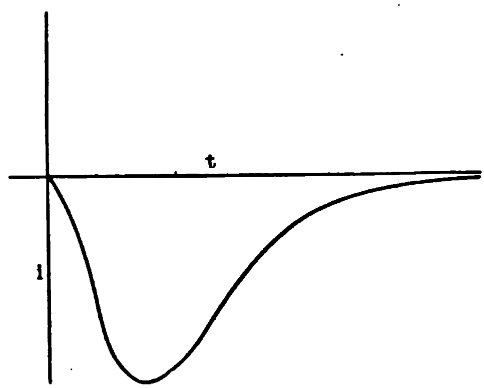


Fig. 109.

a maximum value in the negative direction (corresponding to $\frac{di}{dt}=0$) and then decreases to zero. The shape of the curve represented by equation (40) in this case is as shown in Fig. 109.

*The symbol sinh x is used to represent the expression $\frac{\epsilon^x - \epsilon^{-x}}{2}$, which is called the *hyperbolic sine of x*.

For $r^2 < \frac{4L}{C}$, we have $m=\sqrt{(-1)\left(\frac{1}{LC}-u^2\right)}$ or $m=j\omega$ where $j=\sqrt{-1}$ and $\omega=\sqrt{\frac{1}{LC}-u^2}$, a real quantity. In this case equation (40) may be written

$$i=-\frac{V}{\omega L}\epsilon^{-ut}\sin(\omega t) \qquad (40b)$$

since $\frac{\epsilon^{j\omega t}-\epsilon^{-j\omega t}}{2j}=\sin(\omega t)$. Equation (40*b*) tells us that when $r^2 < \frac{4L}{C}$ the current in the circuit oscillates with a frequency

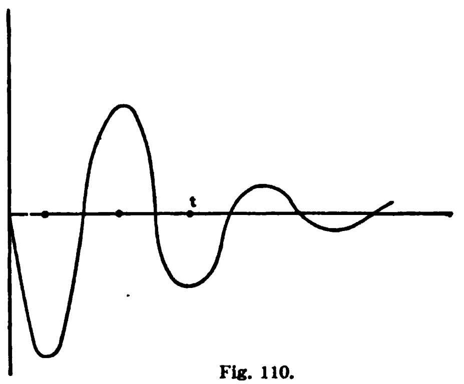


Fig. 110.

$f=\frac{\omega}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{1}{LC}-u^2}$ but that the amplitude of the oscillations decreases as time increases, as shown in Fig. 110.

For $r^2=\frac{4L}{C}$, the constant $m=0$ and therefore $\epsilon^{mt}-\epsilon^{-mt}$ is also zero. Hence equation (40) for the current is indeterminate. Evaluating the indeterminate expression $\frac{\epsilon^{mt}-\epsilon^{-mt}}{m}$ we get

$$\left|\frac{\epsilon^{mt}-\epsilon^{-mt}}{m}\right|_{m=0}=\left|\frac{\frac{d}{dm}(\epsilon^{mt}-\epsilon^{-mt})}{\frac{d}{dm}(m)}\right|_{m=0}=\left|\frac{t(\epsilon^{mt}+\epsilon^{-mt})}{1}\right|_{m=0}=2t.$$

Hence in this case equation (40) becomes

$$i=-\frac{Vt\epsilon^{-ut}}{L} \qquad (40c)$$

which likewise represents a non-oscillating discharge such as shown in Fig. 109. When $r^2=\frac{4L}{C}$ the current just ceases to be oscillatory. Equation (40*c*) may therefore be looked upon as the limiting form of the oscillating current given by (40*b*).

SUMMARY OF IMPORTANT DEFINITIONS AND PRINCIPLES

Note: The definitions given in paragraphs 1 to 9 inclusive are in terms of current; they also apply to electromotive forces and potential differences.

1. An **alternating current** is a current which varies continuously with time from a constant maximum in one direction to an equal maximum in the opposite direction and back again to same maximum in the first direction, repeating this cycle of values over and over again in equal intervals of time.

2. The **period** T of an alternating current is the time taken for the current to pass through a complete cycle of positive and negative values.

3. The **frequency** f of an alternating current is the number of complete cycles of values which it passes through in one second

$$f=\frac{1}{T}$$

4. The number of **alternations** a per minute is the total number of times per minute that the current changes in direction

$$a=120f=\frac{120}{T}$$

5. The equation of a harmonic current of maximum value I_0 is

$$i=I_0 \sin (\omega t+\theta)$$

where ω, called the **periodicity** of the current, is equal to $\frac{2\pi}{T}=2\pi f$, and θ, called the **phase** of the current, is a constant such that $I_0 \sin \theta$ gives the value of the current at time $t=0$.

6. The **difference in phase** between a harmonic current and a harmonic *p.d.* is the angle corresponding to the time between

successive maxima values of the current and *p.d.* respectively.

7. The equation of a **non-harmonic current** of frequency f may be written

$$i = I_1 \sin(\omega t + \theta_1) + I_2 \sin(2\omega t + \theta_2) + I_3 \sin(3\omega t + \theta_3) + etc.$$

where $\omega = 2\pi f$ and the I's and θ's are constants. The first term in this expression is called the fundamental or first harmonic, the succeeding terms the second, third, etc., harmonics.

8. The **instantaneous value** of an alternating current is its value at any instant; the **maximum value** of an alternating current is its greatest instantaneous value during any cycle; the **average value** of an alternating current is the numerical value of the average of its instantaneous values between successive zero values. For a harmonic current

$$I_{aver.} = \frac{2}{\pi} I_{max.}$$

9. The **effective value** of an alternating current is the square root of the mean of the squares of its instantaneous values over a complete period, that is,

$$I_{eff} = \sqrt{\frac{1}{T}\int_0^T i^2\, dt}$$

where T is a complete period of the current and i its instantaneous value at any instant. When an alternating current is expressed as so many amperes this effective value is always meant unless specifically stated otherwise. The effective value of a harmonic current is

$$I_{eff} = \frac{I_{max}}{\sqrt{2}}.$$

10. When a harmonic current of effective value I and a harmonic potential drop of the same frequency and of effective value V exist in a circuit, the **average electric power input** into this circuit during each cycle is

$$P = VI \cos\theta$$

where θ is the difference in phase between the current and *p.d.*

11. The **power factor** of a circuit is the ratio of the average power input P to the product of the effective value V of the *p.d.* by the effective value I of the current, *i.e.*,

$$\text{Power factor} = \frac{P}{VI}$$

12. The **average electric power input** into a circuit when the **current and p.d. are not harmonic** functions of time, or sine waves, is equal sum of the values of the average power corresponding to each pair of harmonics of the *same* frequency, *i.e.*,

$$P=V_1I_1 \cos \theta_1+V_2I_2 \cos \theta_2+V_3I_3 \cos \theta_3+etc.$$

where the *V's* and *I's* are the **effective** values of the successive harmonics of the *p.d.'s* and currents, and the *θ's* the difference in phase between the corresponding harmonics of the same frequency.

13. The **effective value of a non-harmonic current** is equal to the square root of the sum of the squares of the effective values of all the harmonics present in the current wave, *i.e.*, is equal to

$$I=\sqrt{I_1^2+I_2+I_3^2+ etc.}$$

where I_1, I_2, I_3, etc., are the effective values of the harmonics. Similarly the effective value of a non-harmonic *p.d.* is

$$V=\sqrt{V_1^2+V_2^2+V_3^2+etc.}$$

where V_1, V_2, V_3, etc., are the effective values of the harmonics in the *p.d.* wave.

14. The **equivalent sine-wave** p.d. **and current** are the harmonic *p.d.* and the harmonic current which have respectively the same effective values as the actual *p.d.* and current and differ in phase by the angle whose cosine is equal to the power factor.

15. The **form factor** of a wave is the ratio of the effective .value to the average value.

16. The **amplitude factor** of a wave is the ratio of the maximum value to the effective value.

17. The component of the current in any portion of a circuit in phase with the potential drop through this portion of the circuit is called the **power component of the current** in this portion of the circuit. The component of the *p.d.* in phase with the current is called the **power component of the p.d.** When the current and *p.d.* are sine waves having the effective values V and I respectively and differing in phase by the angle θ, the effective value of the power component of the current is $I \cos \theta$ and the effective value of the power component of the *p.d.* is $V \cos \theta$.

The component of the current in quadrature with the *p.d.* is called the **reactive component of the current** and the component of the *p.d.* in quadrature with the current is called the **reactive component of the p.d.** For sine-wave current and *p.d.* the

effective value of the reactive component of the current is $I \sin \theta$ and the effective value of the reactive component of the *p.d.* is $V \sin \theta$.

18. **Apparent power, volt-amperes or apparent watts** is the product of the effective value of the *p.d.* by the effective value of the current.

19. **Reactive power** is the product of the effective value of the current (or *p.d.*) by the effective value of the component of the *p.d.* (or current) in phase with it.

20. A **sine-wave current** (*p.d.* or *e.m.f.*) may be **represented by a vector** equal in length to the effective value of the current (*p.d.* or *e.m.f.*) making an angle with an arbitrarily chosen axis of reference equal to its phase angle.

21. The average power corresponding to a sine-wave *p.d.* and a sine-wave current is equal to the product of the length of the vector representing the *p.d.* by the projection of the vector representing the current on the *p.d.* vector, which in turn is equal to the product of the length of the current vector by the projection of the *p.d.* vector upon the current vector.

22. The effective value of the potential drop due to a sine-wave current of effective value I and frequency f in a circuit having a constant resistance r and a constant inductance L is

$$V = I\sqrt{r^2 + (2\pi f L)^2}$$

and leads the current by the angle

$$\theta = \tan^{-1}\frac{2\pi f L}{r}$$

23. The **effective resistance** r of any portion of a circuit to an alternating current is the ratio of the average rate P_h at which heat energy is developed in this portion of the circuit to the square of the effective value I of the current in this portion of the circuit, *i.e.*,

$$r = \frac{P_h}{I^2}$$

24. The **impedance** z of any portion of a circuit is the ratio of the numerical value of the vector difference of the externally induced *e. m. f.* E and the resultant *p.d.* V in this portion of the circuit, both in the direction of the current, to the effective value I of the current, *i.e.*,

$$z=\frac{\overline{E-V}}{I}$$

When there is no externally induced *e. m. f.*

$$z=\frac{V}{I}$$

25. The **reactance** x of any portion of a circuit is the square root of the difference between the squares of the impedance and the reactance of the given portion of the circuit, *i.e.*,

$$x=\sqrt{z^2-r^2}$$

The reactance is taken positive when the current lags behind the potential drop in the direction of the current, negative when the current leads this *p.d.*

26. In the case of a sine-wave current and no externally induced *e. m. f.* the angle by which the current lags behind the *p.d.* is

$$\theta=tan^{-1}\frac{x}{r}$$

27. The impedance of a circuit of constant resistance r and inductance L to a sine-wave current of frequency f is

$$z=\sqrt{r^2+(2\,\pi\,f\,L)^2}$$

and its reactance is

$$x=2\,\pi\,f\,L$$

28. The impedance of a condenser of capacity C and leakance g to a sine-wave current of frequency f is

$$z=\frac{1}{\sqrt{g^2+(2\,\pi\,fC)^2}}$$

and its reactance is

$$x=-\frac{2\,\pi\,f\,C}{g^2+(2\,\pi\,f\,C)^2}$$

29. The impedance of a circuit formed by a resistance r, an inductance L and a capacity C in series to a sine-wave current is

$$z=\sqrt{r^2+\left(2\,\pi\,f\,L-\frac{1}{2\,\pi\,f\,C}\right)^2}$$

and the reactance is

$$x=2\,\pi\,f\,L-\frac{1}{2\,\pi\,f\,C}$$

provided the condenser of capacity C has no leakance.

30. A circuit is said to be in **resonance** with the impressed electromotive force when the frequency of the impressed *e. m. f.* is the same as the natural or free frequency of the circuit. The free frequency of a closed circuit formed by a capacity C and an inductance L, when the resistance and leakance are negligible, is

$$\frac{1}{2\pi\sqrt{LC}}$$

31. In a circuit formed of two or more impedances in series, the resultant effective resistance is the arithmetical sum of the separate effective resistances, the resultant reactance is the algebraic sum of the separate reactances, and the resultant impedance is the square root of the sum of the squares of the resultant resistance and reactance, provided the current and *p.d.* are sine waves; that is

$$R = r_1 + r_2 + etc.$$
$$X = x_1 + x_2 + etc.$$
$$Z = \sqrt{R^2 + X^2}$$

32. The **effective conductance** g of any part of a circuit to an alternating current is the ratio of the average rate P_h at which heat energy is developed in this portion of the circuit to the square of the effective value V of the *p.d.* through it, *i.e.*,

$$g = \frac{P_h}{V^2}$$

33. The **admittance** y of any part of a circuit is the ratio of the effective value of the current I in this portion of the circuit to the numerical value of the vector difference of the externally induced *e. m. f.* E and the resultant *p.d.* V in this portion of the circuit, both in the direction of the current, *i.e.*,

$$y = \frac{I}{E - V}$$

When there is no externally induced *e. m. f.*

$$y = \frac{I}{V}$$

34. The **susceptance** of any portion of a circuit is square root of the difference between the squares of the impedance and the reactance of this portion of the circuit, *i.e.*,

$$b = \sqrt{y^2 - g^2}$$

The susceptance is taken positive when the current lags behind

the potential drop in the direction of the current, negative when the current leads this *p.d.*

35. In the case of a harmonic current and no externally induced *e. m. f.* the angle by which the current lags behind the *p.d.* is

$$\theta = tan^{-1}\frac{b}{g}$$

36. The resistance r, the conductance g, the reactance x, the susceptance b, the impedance z and the admittance y of a given portion of a circuit when the current and *p.d.* are sine waves are related as follows:

$$z = \sqrt{r^2 + x^2} \qquad y = \sqrt{g^2 + b^2}$$

$$z = \frac{1}{y} \qquad y = \frac{1}{z}$$

$$r = \frac{g}{y^2} \qquad g = \frac{r}{z^2}$$

$$x = \frac{b}{z^2} \qquad b = \frac{x}{z^2}$$

37. The **admittance of a condenser** of capacity C and leakance g to a sine-wave current of frequency f is

$$y = \sqrt{g^2 + (2\pi f C)^2}$$

and its susceptance is

$$b = -2\pi f C$$

38. In a circuit formed of two or more **impedances in parallel,** the resultant effective conductance is the arithmetical sum of the separate effective conductances, the resultant susceptance is the algebraic sum of the separate susceptances, and the resultant admittance is the square root of the sum of the squares of the resultant effective conductance and susceptance, provided the current and *p.d.* are sine waves, that is

$$G = g_1 + g_2 + etc.$$

$$B = b_1 + b_2 + etc.$$

$$Z = \sqrt{G^2 + B^2}$$

39. The discharge of a condenser of capacity C but no leakance through a resistance r and an inductance L in series is oscillatory when

$$r^2 < \frac{4\,L}{C}$$

and non-oscillatory when

$$r^2 > \frac{4\,L}{C}$$

PROBLEMS

1. The equation of an alternating current is $i = 100 \sin 377t$. Find (1) the maximum value of the current, (2) the rate at which the current is changing when the current is a maximum, (3) the rate at which the current is changing when the current is zero, (4) the effective value, (5) the average value, (6) the frequency, (7) the period, and (8) the periodicity.

Ans.: (1) 100 amperes; (2) 0 amperes per second; (3) 37,700 amperes per second; (4) 70.7 amperes; (5) 63.7 amperes; (6) 60 cycles per second; (7) 0.01667 second; (8) 377 radians per second.

2. Find (1) the effective value, (2) the average value, (3) the form factor, and (4) the amplitude factor of a semi-circular shaped wave, the maximum value of which is A.

Ans.: (1) 0.816 A; (2) 0.785 A; (3) 1.04; (4) 1.23.

3. If a direct current of 10 amperes and an alternating current of 10 amperes (effective) exist at the same time in a circuit, what will an alternating current ammeter connected in series with this circuit indicate?

Ans.: 14.14 amperes.

4. The equation of the current in a circuit is

$$i = 50 \sin(\omega t + 20°) + 30 \sin(3\,\omega t - 15°) + 10 \sin(5\,\omega t + 30°)$$

and the equation of the potential drop through it is

$$v = 100 \sin(\omega t - 10°) + 40 \sin(3\,\omega t - 30°)$$

Determine (1) the effective value of the current, (2) the effective value of the potential difference across the circuit, (3) the average power absorbed by the circuit, (4) the power factor of the circuit, and (5) the equivalent sine waves of current and potential difference.

Ans.: (1) 41.8 amperes; (2) 76.1 volts; (3) 2744 watts; (4) 86.3%; (5) $i = 59.2 \sin \omega t$ and $v = 107.8 \sin(\omega t - 30.3°)$.

5. The equations of the potential drop through the three parts of a series circuit are $v_1 = 80 \sin(\omega t + 60°)$, $v_2 = 60 \sin \omega t$ and

$v_3 = 50 \sin(\omega t - 80°)$. Find (1) the equation of the potential drop through the entire circuit, and (2) the effective value of the total potential drop.

Ans.: (1) $v = 110.6 \sin(\omega t + 10.5°)$; (2) 78.2 volts.

6.* An *e. m. f.* of 150 volts, the frequency of which is 60 cycles, is impressed upon a series circuit consisting of a resistance of 5 ohms, an impedance coil of 2 ohms resistance and 0.1 henry inductance and a condenser of 50 microfarads capacity. Find (1) the value of the current established in the circuit and the potential drops, (2) through the 5 ohm resistance, (3) through the impedance, and (4) through the condenser. Find the angle by which the current lags behind the potential drop (5) through the 5 ohm resistance, (6) through the impedance coil, (7) through the condenser and (8) through the entire circuit. Draw a complete vector diagram.

Ans.: (1) 8.88 amperes; (2) 44.4 volts; (3) 336 volts; (4) 472 volts; (5) 0°; (6) 87.0°; (7) −90°; (8) −65.6°.

7. (1) What is the maximum 60-cycle *e. m. f.* which may be impressed upon a circuit formed by an impedance of 0.1 ohm resistance and 0.3 henry inductance and a condenser of 23.45 microfarads capacity connected in series, if the effective puncturing voltage of the condenser dielectric is 10,000 volts? (2) If a 25-cycle *e. m. f.* of the same value as this 60-cycle *e. m. f.* determined above is impressed upon this circuit, what is the potential drop across the condenser?

Ans.: (1) 8.84 volts; (2) 10.69 volts.

8. When an *e. m. f.* of 220 volts is impressed upon a circuit formed by two impedances *A* and *B* connected in series, the power absorbed by *A* is 300 watts and the power absorbed by *B* is 1200 watts. If the potential drop across *A* is 200 volts and the current in *A* is 10 amperes, find (1) the potential drop across *B*, (2) the resistances, and (3) the reactances of *A* and *B* respectively.

Ans.: (1) 125.4 volts; (2) 378 volts; (3) 3 and 12 ohms; (4) ± 19.8 ohms for *A* and ± 3.67 or ± 35.9 ohms for *B*.

9. The current established in a circuit consisting of a resistance of 3 ohms in series with an impedance of unknown resistance

* The values of the *e. m. f.*, potential difference, and current in Problem 6 and in all problems following are effective values and the wave forms sinusoidal.

and reactance is 12 amperes. If the power absorbed by the circuit is 600 watts and the potential drop across the impedance is 100 volts, find (1) the impressed voltage on the circuit, (2) the power factor of the circuit, and (3) the angle by which the drop through the impedance leads the current.

Ans.: (1) 111 volts; (2) 45.1%; (3) 82.0°.

10. Two impedances A and B are connected in parallel. A has a resistance of 5 ohms and an inductance of 0.02 henry, while B is formed by a resistance of 10 ohms and a capacity of 100 microfarads in series. If a 25-cycle *e.m.f.* of 220 volts is impressed upon this parallel circuit find (1) the total current, (2) the current in each branch, and (3) the power factor of the circuit.

Ans.: (1) 36.1 amperes; (2) 37.3 and 3.41 amperes respectively; (3) 89.2%.

11. At 60 cycles the impedances of two coils A and B are each 10 ohms. At 25 cycles the impedance of A is 5.38 ohms and the impedance of B is 8.67 ohms. Find the joint impedance of A and B at 60 cycles when connected (1) in parallel, and (2) in series.

Ans.: (1) 5.24 ohms; (2) 19.06 ohms.

12. A parallel circuit consists of a coil (A) of negligible resistance and 0.05 henry inductance connected in parallel with a condenser (B) of 140.7 microfarads capacity and negligible leakance. This parallel circuit is connected in series with an impedance (C) of 2 ohms resistance and 0.02 henry inductance. Find (1) the current, and (2) the potential drop in A, B and C respectively, when an *e. m. f.* of 220 volts and 60 cycles is impressed upon this circuit. If a non-inductive resistance (D) of 10 ohms is connected in parallel with (A), find (3) the current and (4) the potential drop in A, B, C and D respectively. Draw a vector diagram for each case.

Ans.: (1) 11.67, 11.67 and 0 amperes; (2) 220, 220 and 0 volts; (3) 8.24, 8.24, 15.53 and 15.53 amperes; (4) 155.3, 155.3, 121.0 and 155.3 volts.

13. When a 50-kw., 60-cycle, a.c. generator is delivering 30 kw. at 80% power factor the terminal voltage is 230 volts. The effective resistance of the armature between terminals is 0.05 ohm and the reactance 0.1 ohm. Neglecting the effect of armature reaction, (1) what would be the terminal voltage of this machine if the external circuit were opened? (2) What is the maximum

load under which this generator could operate continuously at 80% power factor?

Ans.: (1) 246.5 volts; (2) 40 kw.

14. 500 kw. are delivered to a substation from a 6600-volt power station over a transmission line of 6 ohms resistance. If the current taken by the load is 100 amperes, find (1) the power factor at the power station, and (2) the efficiency of the transmission line.

Ans.: (1) 84.9%; (2) 89.3%.

15. The potential differences at the load and generator ends of a transmission line are each 6600 volts. The resistance of the line is 6 ohms and the reactance 8 ohms. If the line current is 100 amperes, find (1) the power factor at the load, and (2) the efficiency of the transmission line.

Ans.: (1) 75.2%; (2) 89.2%.

16. A transmission line 25 miles in length consists of two No. 0 B. & S. wires (diameter 0.325 inch) spaced 3 feet apart. At the end of this line is connected a 60-cycle induction motor load (lagging current) of 1500 kw. operating at 25,000 volts and at a power factor of 75%. What must be the potential difference at the power station supplying this load? The leakage and capacity of the line are to be neglected.

Ans.: 28,400 volts.

17. Two 220 alternators *A* and *B* connected in parallel supply an inductive load of 50 kw. at 90% power factor. If *A* supplies one-third of this load at a power factor of 80%, find (1) the power factor at which *B* supplies power to the load, and (2) the armature current of each alternator, if both currents lag behind the generated *e. m. f.'s*.

Ans.: (1) 94.3%; (2) 94.7 and 160 amperes.

VIII

SYMBOLIC METHOD OF TREATING ALTERNATING CURRENTS

203. Symbolic Representation of a Vector.—Let $A = OP$ in Fig. 111 be any vector making an angle θ with an arbitrary line of reference OX. The component of A in the direction of OX is

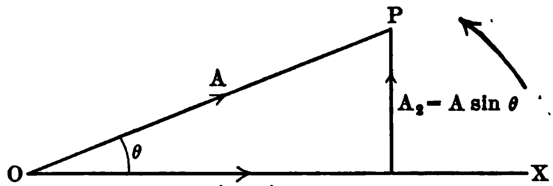


Fig. 111.

then $A \cos\theta$ and the component of A 90° ahead of this line of reference is $A \sin\theta$. Put

$$A \cos\theta = A_1 \qquad (1a)$$
$$A \sin\theta = A_2$$

Employing the usual convention of indicating a *vector* sum we may write $A = \overline{A_1 + A_2}$, where the line over A_1 and A_2 indicates that A_1 and A_2 are to be added vectorially. Instead of using this symbol, however, we may *in the special case of two mutually perpendicular vectors* represent the vector A by the symbolic expression

$$\underset{\cdot}{A} = A_1 + jA_2 \qquad (1b)$$

where the symbol j indicates that the vector A_2 leads the line of reference by 90°, while A_1 coincides in direction with the line of reference.

When a single letter is used to represent a vector in this symbolic notation, it is usual to write a dot under the letter. That is, the letter A by itself represents the length of the vector, while $\underset{\cdot}{A}$ represents the vector expressed in this symbolic notation.

Squaring and adding equations (1a) and taking the square root, we get

$$A = \sqrt{A_1^2 + A_2^2} \qquad (1c)$$

Therefore, the length of a vector is equal to the square root of the sum of the squares of the two components entering into its symbolic expression. Taking the ratio of two equations (1*a*) we get

$$\tan\theta = \frac{A_2}{A_1} \qquad (1d)$$

Whence, the angle which a vector makes with the line of reference is equal to the angle whose tangent is the ratio of the second (or j component) to the first component in its symbolic expression.

Similarly, any other vector $A' = OP'$ making an angle θ' with the same line of reference may be resolved into the two components $A_1' = A' \cos\theta'$ and $A_2' = A' \sin\theta'$, and we may write

$$\dot{A}' = A_1' + jA_2'$$

where j as before indicates that the vector A_2' leads the line of reference by 90° while A_1' coincides in direction with the line of reference.

204. Addition of Vectors.—From Fig. 112 it follows that the resultant of two vectors $\dot{A}$ and $\dot{A}'$ is the vector which has the com-

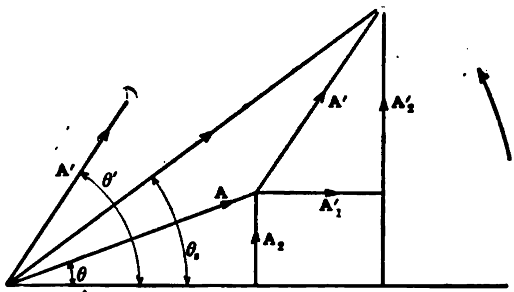


Fig. 112.

ponent $A \cos\theta + A' \cos\theta' = A_1 + A_1'$ parallel to the axis of reference and the component $A \sin\theta + A' \sin\theta' = A_2 + A_2'$ leading the line of reference by 90°. The resultant of the two vectors $\dot{A}$ and $\dot{A}'$ is then represented symbolically as

$$\dot{S} = \dot{A} + \dot{A}' = (A_1 + A_1') + j\,(A_2 + A_2') \qquad (2a)$$

The length of this vector is then

$$S = \sqrt{(A_1 + A_1')^2 + (A_2 + A_2')^2} \qquad (2b)$$

and it makes an angle

$$\theta_s = \tan^{-1} \frac{A_2 + A_2'}{A_1 + A_1'} \qquad (2c)$$

with the axis of reference. Similarly, the sum of any number of vectors is, in the symbolic notation,

$$\underset{\bullet}{S} = \underset{\bullet}{A} + \underset{\bullet}{A'} + \underset{\bullet}{A''} + \text{etc.} = (A_1 + A_1' + A_1'' + \text{etc.}) + j(A_2 + A_2' + A_2'' + \text{etc.})$$

The length of this resultant vector is

$$S = \sqrt{(A_1 + A_1' + A_1'' + --)^2 + (A_2 + A_2' + A_2'' + --)^2}$$

and the angle which it makes with the axis of reference is

$$\theta_s = \tan^{-1} \frac{A_2 + A_2' + A_2'' + ---}{A_1 + A_1' + A_1'' + ---}$$

205. Subtraction of Vectors. — The vector $\underset{\bullet}{A} - \underset{\bullet}{A'}$ (Fig. 113) is similarly the vector which has the component $A \cos \theta - A' \cos \theta'$

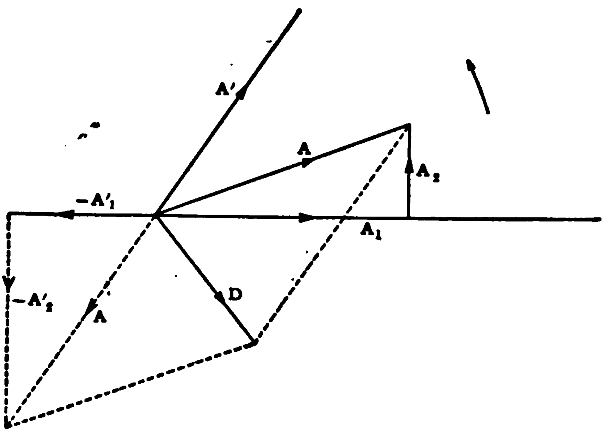


Fig. 113.

$= A_1 - A_1'$ in the direction of the line of reference and the component $A \sin \theta - A' \sin \theta' = A_2 - A_2'$ leading the line of reference by 90°. That is, the vector difference $\underset{\bullet}{A} - \underset{\bullet}{A'}$ is represented symbolically as

$$\underset{\bullet}{D} = \underset{\bullet}{A} - \underset{\bullet}{A'} = (A_1 - A_1') + j(A_2 - A_2') \tag{3a}$$

The length of this vector is then

$$D = \sqrt{(A_1 - A_1')^2 + (A_2 - A_2')^2} \tag{3b}$$

And it makes the angle

$$\theta_d = \tan^{-1} \frac{A_2 - A_2'}{A_1 - A_1'} \tag{3c}$$

with the axis of reference. When this angle comes out negative, the resultant vector makes a negative angle with the axis of

reference or *lags behind* this axis; Fig. 113 illustrates such a case.

206. The Symbol "j" as a Multiplier Signifying Rotation. — So far, we have considered the symbol "j" simply as a means of indicating the component of a vector leading the line of reference. As we shall presently see, it is also frequently convenient to look upon "j" as a multiplier as well as a symbol representing direction, and *to define the operation $j \times \underset{\cdot}{A}$, where $\underset{\cdot}{A}$ is the symbolic expression for any vector, as equivalent to turning the vector $\underset{\cdot}{A}$ through an angle of 90° in the positive direction.* This convention leads to a useful mathematical equivalent for the symbol "j."

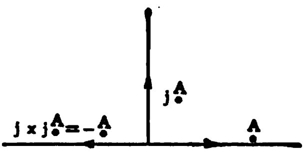


Fig. 114.

Let $\underset{\cdot}{A}$ (Fig. 114) be a vector coinciding in direction with the line of reference. Then from the above convention $j\underset{\cdot}{A}$ represents a vector of length A leading the line of reference by 90°. Multiplying the symbolic expression $j\underset{\cdot}{A}$ for the second vector by j, we have from the above convention that $j \times j\underset{\cdot}{A}$ represents a vector of length A making an angle of 90° with the vector $j\underset{\cdot}{A}$, that is, a vector equal in length to A, but in the opposite direction to $\underset{\cdot}{A}$. Therefore

$$j \times j\underset{\cdot}{A} = -\underset{\cdot}{A}$$

or

$$j^2 = -1$$

whence

$$j = \sqrt{-1}. \tag{4}$$

It should be borne in mind, however, that although the symbol j is mathematically equal to the imaginary quantity $\sqrt{-1}$, the physical meaning of j is not imaginary at all; it is simply an abbreviation placed before the expression for a vector signifying that the vector in question is to be turned through 90° in the positive direction; turning a vector through 90° twice in the same direction is mathematically equivalent to writing a negative sign before the vector. For example, in the expression

$$\underset{\cdot}{A} = A_1 + jA_2$$

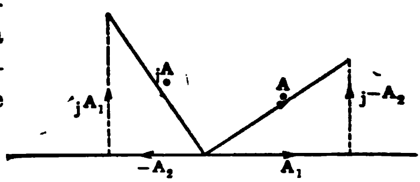


Fig. 115.

for a single vector, A_1 and A_2 both represent vectors coinciding in direction with the line of reference, but in the expression for the resultant vec-

tor $\underset{\cdot}{A}$, the component A_2 is turned through $+ 90°$. Again, when the resultant vector $\underset{\cdot}{A} = A_1 + jA_2$ is turned through 90°, the expression for this vector in its new position is $jA_1 - A_2$, as is readily seen from Fig. 115. But

$$jA_1 - A_2 = j(A_1 + jA_2)$$

since $j^2 = -1$.

207. Symbolic Expression for a Vector Referred to Any Other Vector as the Line of Reference. — Let the vector A' (Fig. 116) lead the vector A by θ degrees; it is desired to write the symbolic expression for A' referred to the vector A as the line of reference. The component of A' parallel to A is $A' \cos \theta$ and the component of A' leading A by 90° is $A' \sin \theta$. Hence, the symbolic expression for A' referred to A as the line of reference is

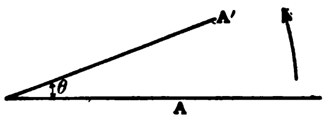


Fig. 116.

$$\underset{\cdot}{A'} = A' (\cos \theta + j \sin \theta) \qquad (5a)$$

Similarly, the symbolic expression for the vector A referred to A' as the line of reference is

$$\underset{\cdot}{A} = A (\cos \theta - j \sin \theta) \qquad (5b)$$

since A *lags behind* A' by the angle θ.

For example, let the vector A' have the length 10 and let this vector *lead* a second vector A by 30°. Then the symbolic expression of A' referred to A as the axis of reference is

$$\underset{\cdot}{A'} = 10 (\cos 30° + j \sin 30°)$$
$$= 8.65 + j\,5$$

If the vector A' *lags behind* the vector A by 30°, then its symbolic expression referred to A as the axis of reference is

$$\underset{\cdot}{A'} = 10 (\cos 30° - j \sin 30°)$$
$$= 8.65 - j\,5$$

208. Difference in Phase Between Two Vectors Expressed in Symbolic Notation.—When the symbolic expression for a vector A referred to any given line of reference is

$$\underset{\cdot}{A} = A_1 + jA_2$$

the angle by which this line *leads* the line of reference is

$$\theta = \tan^{-1} \frac{A_2}{A_1}$$

Similarly, when the symbolic expression for any other vector A' is

$$\underset{\cdot}{A'} = A_1' + jA_2'$$

the angle by which this vector *leads* the line of reference is

$$\theta' = tan^{-1}\frac{A_2'}{A_1'}$$

Hence the angle by which A' *leads* A is

$$\theta' - \theta = tan^{-1}\frac{A_2'}{A_1'} - tan^{-1}\frac{A_2}{A_1} \qquad (6c)$$

or

$$tan(\theta' - \theta) = \frac{A_1 A_2' - A_1' A_2}{A_1 A_1' + A_2 A_2'}$$

For example, let $A = 8 + j\,6$ and $A' = 5 + j\,4$. Then A' leads A by the angle $\theta' - \theta$ where

$$tan\ (\theta' - \theta) = \frac{8\times4 - 5\times6}{8\times5 + 6\times4} = 0.0312$$

$$\theta' = tan^{-1}\frac{4}{5} = 38°\ 40'$$

Therefore A' leads A by

$$\theta' - \theta = 1.8°$$

209. Symbolic Representation of a Harmonic Function. — We have seen (Articles 182 and 186) that a harmonic alternating current $i = \sqrt{2}\,I\,sin\,(\omega t + \theta)$, where I is the effective value of the current, may be written

$$i = \sqrt{2}\,I\,cos\,\theta\,sin\,\omega t + \sqrt{2}\,I\,sin\,\theta\,cos\,\omega t$$

and that the terms $\sqrt{2}\,I\,cos\,\theta\,sin\,\omega t$ may be represented by a vector equal in length to $I_1 = I\,cos\,\theta$ rotating about a fixed point O with the angular velocity ω and the term $\sqrt{2}\,I\,sin\,\theta\,cos\,\omega t$ by a second vector of length $I_2 = I\,sin\,\theta$, leading the first vector by 90° and rotating about this same point with the same angular velocity. Hence the function $i = \sqrt{2}\,I\,sin\,(\omega t + \theta)$ may be represented by a vector

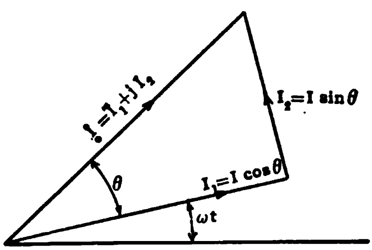


Fig. 117.

$$I = I_1 + jI_2 \qquad (7)$$

rotating about this point with the angular velocity ω, where I_1 is a vector of length $I\,cos\,\theta$, and jI_2 a vector of length $I\,sin\,\theta$ leading I_1 by 90°, both rotating about the same point with the same angular velocity ω. Therefore we may look upon equation (7) as the symbolic expression for the function $i = \sqrt{2}\,I\,sin\,(\omega t + \theta)$. Note, however, that the two vectors I_1 and jI_2 though constant in length *are continually changing in direction;* therefore in the expression

$$\dot{I} = I_1 + jI_2$$

both I_1 and I_2 and therefore $\dot{I}$ also, must be looked upon as functions of time and not as constants, since a vector by definition is a quantity having direction as well as magnitude.

Note the analogy with a body moving in a circle of radius a with a constant linear speed s; the velocity of such a body is constant in magnitude, but is continually changing in direction. Hence the rate of change of the velocity is not zero, but is $\frac{s^2}{r}$, and the direction of this rate of change, *i.e.*, of the acceleration, is perpendicular to the direction of the velocity.

210. Symbolic Expression for the Derivative of a Harmonic Function. — Since a harmonic function of the form

$$i = \sqrt{2}\, I \sin (\omega t + \theta)$$

may be written in the form

$$i = \sqrt{2}\, I \cos \theta \sin \omega t + \sqrt{2}\, I \sin \theta \cos \omega t$$

the derivative of i with respect to time is

$$\frac{di}{dt} = \sqrt{2}\, \omega I \cos \theta \cos \omega t - \sqrt{2}\, \omega I \sin \theta \sin \omega t$$

But $\sqrt{2}\, \omega I \cos \theta \cos \omega t$ is represented by a vector equal in length to $\omega I \cos \theta = \omega I_1$ leading by 90° the vector I_1 which represents $\sqrt{2}\, I \cos \theta \sin \omega t$; and $-\sqrt{2}\, \omega I \sin \theta \sin \omega t$ is represented by a vector of length $\omega I \sin \theta = \omega I_2$ leading by 90° the vector jI_2 which represents $\sqrt{2}\, I \sin \theta \cos \omega t$. Hence the derivative of the function represented by the vector

$$\dot{I} = I_1 + jI_2$$

is

$$\frac{d\dot{I}}{dt} = j\omega (I_1 + jI_2) = j\omega \dot{I} \qquad (8)$$

That is, the derivative with respect to time of a vector rotating with a velocity ω results in a vector equal in length to the product of ω by the length of the original vector, which vector has a direction 90° *ahead of* the original vector.

Vice versa, from equation (8) we also have that

$$j\dot{I} = \frac{1}{\omega} \frac{d\dot{I}}{dt} \qquad (8a)$$

that is, multiplying a rotating vector by j is equivalent to differentiating the vector with respect to time and dividing by ω.

From this signification of the multiplier j it is at once evident that to consider the multiplication of the expressions representing two *rotating* vectors at right angles to each other, such as I and jV as equivalent to $j\,(IV)$ is inconsistent with j as being equivalent to the operation $\frac{1}{\omega}\frac{d}{dt}$. For $\frac{d}{dt}\,(IV) = I\frac{dV}{dt} + V\frac{dI}{dt}$, and therefore to be consistent with equation (8a), the expression $j\,(IV)$ must equal $IjV + VjI$.

Consequently, although the product of the two expressions I_1+jI_2 and V_1+jV_2 representing the current and voltage in a circuit is $I_1V_1 - I_2V_2 + j\,(I_1V_2+I_2V_1)$, when j is considered simply as a multiplier equal to $\sqrt{-1}$, this expression, however, is inconsistent with the meaning of j, as defined by equation (8a), since I_1, I_2, V_1 and V_2 are all *rotating* vectors. It is important to bear this clearly in mind, since one is likely to make the mistake of taking this product as the symbolic expression for the power corresponding to the current I_1+jI_2 and the $p.d.$ V_1+jV_2, and to take the "real" part of this product as representing the average power. As a matter of fact, the value of the average power corresponding to the current I_1+jI_2 and the $p.d.$ V_1+jV_2 is

$$P = I_1V_1 + I_2V_2$$

(see Article 215). That is, the sign between the two terms in the expression for the average power is just the opposite of the sign resulting from the multiplication of I_1+jI_2 and V_1+jV_2 and taking j as a multiplier equal to $\sqrt{-1}$.

211. Symbolic Notation for Impedance. — Impedance as a Complex Number.— When the current and $p.d.$ in a circuit of *constant* resistance r and *constant* inductance L are *both harmonic functions of the same frequency* they may be represented respectively by the two vectors I and V.

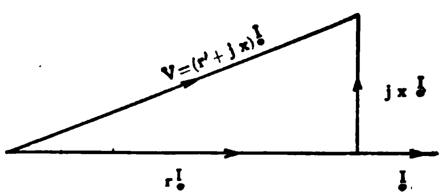


Fig. 118.

The component of the resultant $p.d.$ in phase with the current is a vector coinciding in direction with I and equal in length to the product of r by the length of the current vector, that is, is represented by the vector rI in Fig. 118. The component of the resultant $p.d.$ 90° ahead of the vector I is a vector leading the current vector by

$90°$ and equal in length to the product of the reactance x by the length of the current vector, that is, is represented by the vector jxI. Hence the resultant *p.d.* is represented by the vector

$$\underset{\cdot}{V}=r\underset{\cdot}{I}+j\,x\,\underset{\cdot}{I}=(r+jx)\,\underset{\cdot}{I} \qquad (9)$$

This relation also follows directly from the differential equation

$$\underset{\cdot}{V}=r\underset{\cdot}{I}+L\frac{d\underset{\cdot}{I}}{dt}=r\underset{\cdot}{I}+Lj\,\omega\underset{\cdot}{I}=r\underset{\cdot}{I}+j\,x\,\underset{\cdot}{I}=(r+jx)\,\underset{\cdot}{I}$$

for, since L is a constant, $Lj\,\omega=j\,\omega\,L=jx$.

When the current vector is referred to any arbitrary line of reference rotating with the same angular velocity as the current vector, the symbolic expression for the current vector is $\underset{\cdot}{I}=I_1+jI_2$ where I_1 is the component of $\underset{\cdot}{I}$ parallel to this line of reference and I_2 the component of $\underset{\cdot}{I}$ leading this line of reference by $90°$. The component I_1 flowing through the resistance r produces a drop of potential rI_1 in the phase with I; this component I_1 flowing through the reactance x also produces a drop of potential xI_1 leading I_1 by $90°$. The component I_2 flowing through the resistance r produces a drop of potential rI_2 in phase with I_2 and therefore leading I_1 by $90°$, since I_2 leads I_1 by $90°$; this component I_2 flowing through the reactance x also produces a drop of potential xI_2 leading I_2 by $90°$ and therefore leading I_1 by $180°$. Hence the total drop of potential *in phase* with I_1 is (rI_1-xI_2) and the total drop of potential $90°$ ahead of I_1 is (xI_1+rI_2). Therefore, in symbolic notation the vector representing the total potential drop, referred to I_1 as the line of reference, is

$$\underset{\cdot}{V}=(rI_1-xI_2)+j\;(xI_1+rI_2)$$

But this expression is exactly the same as would result from multiplying (I_1+jI_2) by $(r+jx)$ and considering j as equivalent to $\sqrt{-1}$, that is,

$$\underset{\cdot}{V}=(r+jx)\;(I_1+jI_2) \qquad (9a)$$

This relation also agrees with the interpretation of j as equivalent to a differentiation with respect to ωt, for since r and x are *constants*, it is immaterial whether we write the differentiation sign before or after these constants. Compare with the signification of the operation represented by the formula $(V_1+jV_2)\,(I_1+jI_2)$ where both terms in *each* expression are *rotating* vectors, Article 210.

The expression $r+jx$ is not a vector, although it has the same

mathematical form as the symbolic expression for a vector. Its properties in any operation of multiplication or division are exactly the same as the properties of a "complex number," that is, a number which consists of the sum of two terms, one real and the other imaginary. Since the product of the symbolic expression for current by this expression $r+jx$ gives the symbolic expression for the *p.d.* due to this current in the impedance having a resistance r and reactance x, the expression $r+jx$ may be looked upon as the symbolic expression for impedance, and may be represented by the single symbol z with a dot under it, *i.e.*,

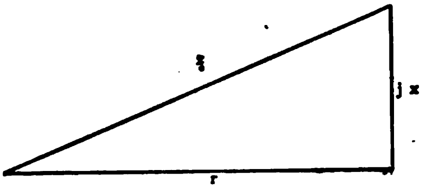


Fig. 119.

$$\underset{\cdot}{z}=r+jx \tag{10}$$

We may then write the symbolic expression for the *p.d.* produced by a current $\underset{\cdot}{I}=I_1+jI_2$ in an impedance $\underset{\cdot}{z}=r+jx$ as

$$\underset{\cdot}{V}=\underset{\cdot}{z}\underset{\cdot}{I} \tag{11}$$

Note that the length of the vector represented by $\underset{\cdot}{z}\underset{\cdot}{I}$ is equal to Iz and that this vector leads the vector $\underset{\cdot}{I}$ by the angle $\theta=tan^{-1}\frac{x}{r}$. Hence multiplying the vector $\underset{\cdot}{I}$ by the impedance $\underset{\cdot}{z}$ gives rise to a vector equal in length to the product of the length of the vector $\underset{\cdot}{I}$ by the numerical value of the impedance, which vector leads $\underset{\cdot}{I}$ by the power-factor angle of the impedance.

212. Symbolic Notation for Admittance. — The relation between current and *p.d.* when *both are harmonic functions of the same frequency* may also be expressed in terms of the admittance of the circuit. Let g be the conductance and b the susceptance, and let $\underset{\cdot}{I}$ and $\underset{\cdot}{V}$ be the current and *p.d.* vectors. Then the component of $\underset{\cdot}{I}$ in phase with $\underset{\cdot}{V}$ is a vector coinciding in direction with the vector $\underset{\cdot}{V}$ and equal in length to the product of the length of the *p.d.* vector by the conductance, that is, is represented by the vector of $g\underset{\cdot}{V}$ in Fig. 120. The component of the current lagging behind the *p.d.* by 90° is a vector lagging behind the vector $\underset{\cdot}{V}$ by 90° and equal in length to the product of the length

of the *p.d.* vector by the susceptance, that is, is represented by the vector $-jb\dot{V}$; since $+j$ represents a lead of 90° and there-

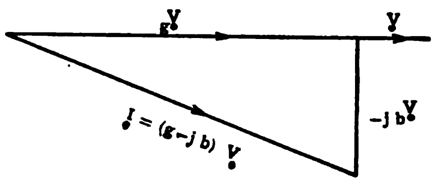


Fig. 120.

fore $-j$ represents a lag of 90°. Hence the resultant current vector is

$$\dot{I}=g\dot{V}-jb\dot{V}=(g-jb)\dot{V} \qquad (12)$$

In the particular case of a leaky condenser this relation also follows from the differential equation

$$\dot{I}=g\dot{V}+C\frac{d\dot{V}}{dt}=g\dot{V}+Cj\omega\dot{V}=(g-jb)\dot{V}$$

for since C is a constant $Cj\omega\dot{V}=j\omega C\dot{V}=-jb\dot{V}$. In the case of a condenser the charging current $\omega C\dot{V}$ leads the *p.d.* by 90°, and therefore the susceptance of the condenser is $-\omega C$, since the susceptance of the circuit is the factor by which the *p.d.* must be multiplied to give the component of the current *lagging* 90° *behind* the *p.d.*

When the *p. d.* is expressed as V_1+jV_2 the operation represented by $(g-jb)(V_1+jV_2)$ can also be shown by a process of reasoning similar to that employed in Article 211 to be equivalent to the product of $g-jb$ and V_1+jV_2 when j is considered as equivalent to $\sqrt{-1}$. Therefore in symbolic notation the admittance of a circuit may be written as a complex number

$$\dot{y}=g-jb \qquad (13)$$

and the current due to a *p.d.* $\dot{V}$ across this admittance may be written

$$\dot{I}=\dot{y}\dot{V} \qquad (14)$$

213. Division of a Rotating Vector by a Complex Number. — Since by equations (11) and (14) we have

$$\dot{V}=\dot{z}\dot{I}=(r+jx)\dot{I} \qquad (a)$$

and

$$\dot{I}=\dot{y}\dot{V}=(g-jb)\dot{V} \qquad (b)$$

the division of the vector $\dot{V}$ by $(r+jx)$ must be equivalent to multiplying the vector $\dot{V}$ by $(g-jb)$, that is

$$\frac{\dot{V}}{\dot{z}}=\dot{y}\dot{V} \qquad \text{or} \qquad \frac{1}{\dot{z}}=\dot{y} \tag{15}$$

Similarly, the division of the vector $\dot{I}$ by $(g-jb)$ must be equivalent to multiplying $\dot{I}$ by $(r+jx)$, that is

$$\frac{\dot{I}}{\dot{y}}=\dot{z}\dot{I} \qquad \text{or} \qquad \frac{1}{\dot{y}}=\dot{z} \tag{16}$$

But multiplying equation (*a*) by $(r-jx)$, considering j simply as a multiplier equal to $\sqrt{-1}$, we get

$$(r-jx)\ \dot{V}=(r^2+x^2)\ \dot{I}$$

whence, dividing this equation by r^2+x^2, we get

$$\dot{I}=\left[\frac{r}{r^2+x^2}-j\frac{x}{r^2+x^2}\right]\dot{V}$$

which to be consistent with $\dot{I}=(g-jb)\ \dot{V}$ requires that

$$g=\frac{r}{r^2+x^2} \text{ and } b=\frac{x}{r^2+x^2}$$

which agree with the relations already established in Article 199.

Similarly, multiplying equation (*b*) by $(g+jb)$ we get

$$(g+jb)\ \dot{I}=(g^2+b^2)\ \dot{V}$$

whence, dividing this equation by r^2+x^2,

$$\dot{V}=\left[\frac{g}{g^2+b^2}+j\frac{b}{g^2+b^2}\right]\dot{I}$$

which to be consistent with $\dot{V}=(r+jx)\ \dot{I}$ requires that

$$r=\frac{g}{g^2+b^2} \text{ and } x=\frac{b}{g^2+b^2}$$

which agree with the relations already established in Article 199.

In general, the division of any rotating vector $\dot{A}$ by a complex number of the form $\dot{a}=a_1\pm ja_2$ is equivalent to multiplying the rotating vector by $\dfrac{a_1\mp ja_2}{a_1^2+a_2^2}$, that is

$$\frac{\dot{A}}{a_1\pm ja_2}=\frac{a_1\mp ja_2}{a_1^2+a_2^2}\dot{A} \tag{17}$$

Consequently, impedance and admittance in the symbolic notation may be treated as algebraic multipliers or divisors, the symbol j in each case being considered as mathematically equivalent to

$\sqrt{-1}$, and whenever a "j" term occurs in the denominator of a fraction, the fraction may, by the relation given by equation (17), be put in a form having a "j" term only in the numerator. Hence, an expression of the form $\frac{\underset{\cdot}{V}}{a_1 \pm a_2}$, where $\underset{\cdot}{V}$ is any rotating vector and $a_1 \pm ja_2$ is any complex number, represents a vector having the component $\frac{a_1 \underset{\cdot}{V}}{a_1^2 + a_2^2}$ in phase with $\underset{\cdot}{V}$ and the component $\mp \frac{a_2 \underset{\cdot}{V}}{a_1^2 + a_2^2}$ leading $\underset{\cdot}{V}$ by 90°.

From equations (15) and (16) the equivalent impedance $\underset{\cdot}{Z}$ of two impedances $\underset{\cdot}{z_1}$ and $\underset{\cdot}{z_2}$ in parallel when there is no externally induced *e. m. f.* in either is

$$\underset{\cdot}{Z} = \frac{\underset{\cdot}{z_1}\underset{\cdot}{z_2}}{\underset{\cdot}{z_1} + \underset{\cdot}{z_2}}$$

For, the admittances corresponding to $\underset{\cdot}{z_1}$ and $\underset{\cdot}{z_2}$ are $\underset{\cdot}{y_1} = \frac{1}{\underset{\cdot}{z_1}}$ and $\underset{\cdot}{y_2} = \frac{1}{\underset{\cdot}{z_2}}$; whence the equivalent admittance is $\underset{\cdot}{Y} = \underset{\cdot}{y_1} + \underset{\cdot}{y_2} = \frac{1}{\underset{\cdot}{z_1}} + \frac{1}{\underset{\cdot}{z_2}} = \frac{\underset{\cdot}{z_1} + \underset{\cdot}{z_2}}{\underset{\cdot}{z_1}\,\underset{\cdot}{z_2}}$; but $\underset{\cdot}{Z} = \frac{1}{\underset{\cdot}{Y}} = \frac{\underset{\cdot}{z_1}\,\underset{\cdot}{z_2}}{\underset{\cdot}{z_1} + \underset{\cdot}{z_2}}$. In any numerical example this expression is readily "rationalized" by applying equation (17). Compare with the formula in Article 98 for two resistances in parallel when there is no electromotive force in either branch.

214. Kirchhoff's Laws in Symbolic Notation. — Kirchhoff's two laws for the relations between the instantaneous values of the current and electromotive force in any circuit, may then be expressed *in symbolic notation* as follows:

1. The sum of all the currents flowing to any point in any network of conductors is zero. That is, at any point

$$\underset{\cdot}{I} + \underset{\cdot}{I'} + \underset{\cdot}{I''} - - - - - - = 0 \qquad (18a)$$

where the currents are all expressed in symbolic notation and are all referred to the *same* line of reference.

2. The sum of all the impedance drops in a given direction around any closed loop in any network of conductors is equal to the sum of all the *externally induced e.m.f.'s* (see p. 339) acting in this loop in this direction. That is, around any closed loop

$$\underset{\cdot}{z}\underset{\cdot}{I} + \underset{\cdot}{z'}\underset{\cdot}{I'} + \underset{\cdot}{z''}\underset{\cdot}{I''} + - - - = \underset{\cdot}{E} + \underset{\cdot}{E'} + \underset{\cdot}{E''} \qquad (18b)$$

where the currents, impedances, and electromotive forces are all expressed in symbolic notation, and the *currents and e.m.f.'s are all referred to the same axis of reference.* That is, the currents are to be expressed as $I=I_1+jI_2$, $I'=I_1'+jI_2'$, etc. and the *e.m.f.'s* as $E=E_1+jI_2$, $E'=E_1'+jE_2'$, etc. where all the components of the currents and *e.m.f.'s* with the subscript 1 are parallel to one another and all those with the subscript 2 are parallel to one another and lead the first set of components by 90°.

The electromotive forces due to inductance and capacity are taken account of by the impedance; the electromotive forces represented by the E's in equation (18*b*) are the externally induced electromotive forces such as those due generators or motors or to the mutual inductance of the two windings of a transformer.

In applying equations (18) to the calculation of the currents and *p.d.'s* in any network of circuits, care must be taken to designate clearly the sense of the vectors representing the currents and *e. m. f.'s.* This is most conveniently done by numbering all the junction points in the network, and designating each current and *e. m. f.* by a double subscript written in the order corresponding to the assumed direction of the current or *e. m. f.* vector. For example, in Fig. 121, the *e.m.f.* from 0 to 1 is represented by E_{01} while the *e. m. f.* from 1 to 0, which is equal to $-E_{01}$ is represented by E_{10}. In the figure, then, the net electromotive force from 1 to 2 is

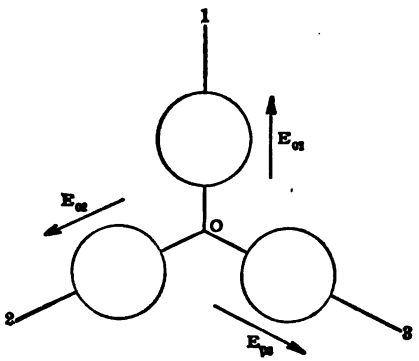


Fig. 121.

$$E_{12}=E_{10}+E_{02}$$

or

$$E_{12}=-E_{01}+E_{02}$$

Each equation of the form (18*a*) or (18*b*) is in reality equivalent to two equations, since the sum of all the "real" terms on one side must be equal to the sum of all the real terms on the other side, and similarly, the sum of all the "j" terms on one side must be equal to the sum of all the "j" terms on the other

side; the denominators of all fractions having been cleared of "j" terms by the transformation given by equation (17). This is merely another way of stating the fact that the component in any direction of the *resultant* of any number of vectors must be equal to the algebraic sum of the components in this direction of all the individual vectors. Applying these two laws to any network enables one, therefore, to calculate both components of every *p.d.* and every current, when the impedances and the electromotive forces are known.

It should be clearly borne in mind that the above equations are true only when the currents, the *e.m.f.'s*, and the impedances are expressed as vector quantities. These equations are not true when the numerical values of these quantities are employed. The indications of alternating current voltmeters and voltmeters give only the *effective* values of currents and *p. d.'s* respectively, they do not indicate their phase relations. Hence, *the sum of the currents entering a junction as measured by ammeters in the various branches is not necessarily zero;* again, *the sum of the potential drops in the various branches of a closed loop, as measured by voltmeters in the various branches, is not necessarily zero.*

Again, these equations hold only when *the currents and e.m.f.'s are all simple harmonic functions* of the *same* frequency *and the resistances and reactances are constant.* When, however, the resistances, inductances and capacities are constant, a similar set of equations holds for each frequency that may be present. Since the equations are all linear in the $\dot{I}$'s and $\dot{E}$'s, the currents and *e.m.f.'s* of any given frequency will be uninfluenced by the presence of currents or *e.m.f.'s* of any other frequency. Hence, when the harmonics present in each *e.m.f.* are known, the harmonics present in each current may be calculated by solving the equations corresponding to the frequency of this particular harmonic, these equations being exactly the same as would hold were all the other harmonics absent.

Note particularly that the above equations do not hold for transient currents; they apply only after the transient terms (see Article 201) have become zero.

215. Expression for Average Power in Symbolic Notation. — When the current in any circuit is $i=\sqrt{2}\,I\,\sin\,(\omega t+\theta)$ and the potential drop in the circuit (in the direction of the current) is

$v=\sqrt{2}\ V\ sin\ (\omega t+\theta')$ the average power input into the circuit is

$$P=VI\ cos\ (\theta'-\theta)$$

where V and I are the effective values of the current and *p.d.* (see equation 9 of Chapter VII). The current $i=\sqrt{2}\,I\ sin(\omega t+\theta)$ may also be written

$$i=\sqrt{2}I\ cos\ \theta\ sin\ \omega t+\sqrt{2}\ I\ sin\ \theta\ cos\ \omega t$$

and may therefore be represented by the vector

$$\dot{I}=I_1+jI_2$$

where

$$I_1=I\ cos\ \theta$$
$$I_2=I\ sin\ \theta$$

Similarly, the *p.d.* $v=\sqrt{2}\ V\ sin\ (\omega t+\theta')$ may be written

$$v=\sqrt{2}\ V\ cos\ \theta'\ sin\ \omega t+\sqrt{2}\ V\ sin\ \theta'\ cos\ \omega t$$

and may therefore be represented by the vector

$$\dot{V}=V_1+jV_2$$

where

$$V_1=V\ cos\ \theta_1$$
$$V_2=V\ sin\ \theta_1$$

Expanding the power equation we get

$$P=VI\ cos\ \theta\ cos\ \theta'+VI\ sin\ \theta\ sin\ \theta'$$
$$=V\ cos\ \theta'\times I\ cos\ \theta+V\ sin\ \theta'\times I\ sin\ \theta$$

But $V\ cos\ \theta'=V_1$; $I\ cos\ \theta=I_1$; $V\ sin\ \theta'=V_2$; $I\ sin\ \theta=I_2$.
Hence the average power corresponding to $\dot{I}=I_1+jI_2$ and $\dot{V}=V_1+jV_2$ is

$$P=V_1I_1+V_2I_2 \qquad (19)$$

Note that this expression is *not* equal to the real part of the product of $\dot{V}=V_1+jV_2$ and $\dot{I}=I_1+jI_2$, when "j" is considered equivalent to $\sqrt{-1}$, which product gives

$$\dot{V}\dot{I}=V_1I_1-V_2I_2+j\,(V_1I_2+V_2I_1).$$

The average power is the real part of this product with the sign between the two terms V_1I_1 and V_2I_2 reversed.

216. Expression for Reactive Power in Symbolic Notation. — The reactive power corresponding to the current $i=\sqrt{2}I\,sin(\omega t+\theta)$ and the *p.d.* $v=\sqrt{2}\ V\ sin\ (\omega t+\theta')$ is (see Article 184)

$$U=VI\ sin\ (\theta'-\theta)$$

(Note that the reactive power is taken as positive when the current lags behind the *p.d.*, *i.e.*, when $\theta<\theta'$.)

Expanding the equation for U we get

$$U = VI \sin \theta' \cos \theta - VI \cos \theta' \sin \theta$$
$$= V \sin \theta' I \cos \theta - V \cos \theta' I \sin \theta$$

Therefore

$$U = V_1 I_2 - V_2 I_1 \tag{20}$$

When I_1, V_1, I_2 and V_2 have the same signification as in the preceding article. Therefore the reactive power is not equal to the "j" part of the product $V = V_1 + jV_2$ and $I = I_1 + jV_2$ when j is considered equivalent to $\sqrt{-1}$, but is the "j" part of this product *with the sign between the terms* $V_1 I_2$ *and* $V_2 I_1$ *reversed.*

217. Expression for Power Factor in Symbolic Notation. — The power factor corresponding to the current $I = I_1 + jI_2$ and the *p.d.* $V = V_1 + jV_2$ is equal to the ratio of the average power $P = V_1 I_1 + V_2 I_2$ to the product of the effect current I and the effective *p.d.* V, that is

$$P.F. = \frac{W}{VI} = \frac{V_1 I_1 + V_2 I_2}{\sqrt{(V_1^2 + V_2^2)\,(I_1^2 + I_2^2)}} \tag{21}$$

The power-factor angle, *i.e.*, the angle the cosine of which is equal to the power factor, is $\theta' - \theta$, since $\theta' - \theta$ is the difference in phase between current and *p.d.* Therefore from (19) and (20)

$$P.F.\ Angle = tan^{-1}\left[\frac{V_1 I_2 - V_2 I_1}{V_1 I_1 + V_2 I_2}\right] \tag{21a}$$

218. Examples of the Use of the Symbolic Method.— Problem I, Series Circuits. — An impedance z_1 has a resistance of 3 ohms and an inductive reactance of 4 ohms; a second impedance z_2 has

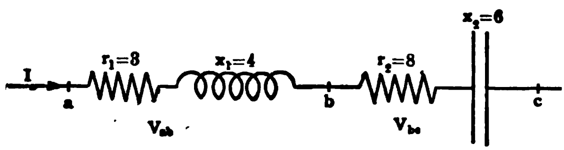


Fig. 122.

a resistance of 8 ohms and a capacity reactance of 6 ohms. z_1 is then represented symbolically as

$$z_1 = 3 + j4$$

and z_2 as

$$z_2 = 8 - j6$$

Let these two impedances be connected in series, Fig. 122, and let an *e. m. f.* of 100 volts be impressed across them from *a* to *c*. Choosing the vector representing the potential drop from *a* to *c*

as the axis of reference, and calling the current from a to c $\dot{I}_{ac}$, we have, that $(3+j4+8-j6)\ \dot{I}_{ac}=100+j0$ whence

$$\dot{I}_{ac}=\frac{100}{11-2j}=\frac{1100+j200}{121+4}=8.8+j1.6$$

Hence the effective value of the current is

$$I=\sqrt{(8.8)^2+(1.6)^2}=8.94 \text{ amperes}$$

and it leads the potential drop from a to c by the angle

$$\tan^{-1}\frac{1.6}{8.8}=10.3°$$

The potential drop across the first impedance is

$$\dot{V}_{ab}=(3+j4)\ (8.8+j1.6)=26.4-6.4+j\ (35.2+4.8)$$
$$=20+j40$$

which has the effective value

$$V_{ab}=\sqrt{(20)^2+(40)^2}=44.7 \text{ volts}$$

and leads the potential drop $\dot{V}_{ac}$ by the angle

$$\tan^{-1}\frac{40}{20}=63.5°$$

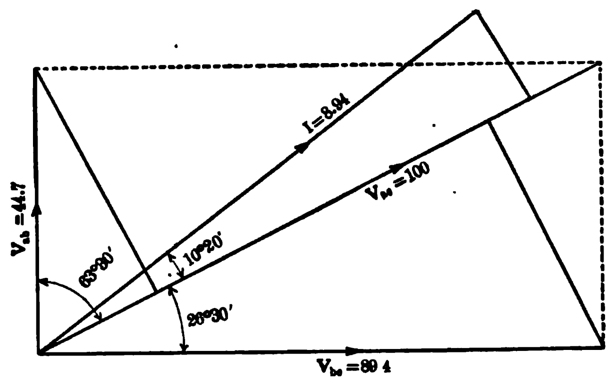


Fig. 123.

The potential drop across the second impedance is

$$\dot{V}_{bc}=(8-j6)\ (8.8+j1.6)=70.4+9.6-j\ (52.8-1.28)$$
$$=80-j40$$

which has the effective value

$$V_{bc}=\sqrt{(80)^2+(40)^2}=89.4 \text{ volts}$$

and lags behind the potential drop $\dot{V}_{ac}$ by the angle

$$\tan^{-1}\frac{40}{80}=26.5°$$

The complete vector diagram is given in Fig. 123. Note that the current is plotted to ten times the scale of the *p.d.*

The power input into the first impedance is

$$W_{ab}=8.8\times 20+1.6\times 40=240 \text{ watts}$$

The power input into the second impedance is

$$W_{bc}=8.8\times 80-1.6\times 40=640 \text{ watts}$$

The total power input is

$$W_{ac}=8.8\times 100+1.6\times 0=880 \text{ watts}$$

which of course is the sum of W_{ab} and W_{bc}.

Problem 2. Parallel Circuits.

Next, let these two impedances be connected in parallel, Fig. 124, and let the total current taken by the two be 100 amperes.

Choosing the vector representing the total current from *a* to *b* as the axis of reference, and calling $I'_{ab}=I_1'+jI_2'$ the current in No. 1 from *a* to *b* and $I''_{ab}=I_1''+jI_2''$ the current in No. 2 from *a* to *b*, we have from equations (18) that

$$I_1'+jI_2'+I_1''+jI_2''=100+j\,0$$

and

$$(3+j4)\,(I_1'+jI_2')-(8-j6)\,(I_1''+jI_2'')=0$$

Whence

$$I_1'+I_1''+j\,(I_2'+I_2'')=100+j\,0$$

$$3I_1'-4I_2'-j\,(4I_1'+3I_2')=8I_1''+6I_2''+j\,(8I_2''-6I_1'')$$

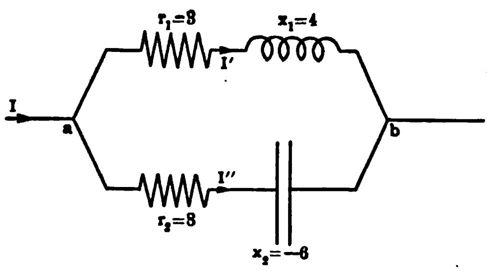


Fig. 124.

Equating the real and the "j" terms in these two equations we get

$$I_1'+I_1''=100$$

$$I_2'+I_2''=0$$

$$3I_1'-4I_2'=8I_1''+6I_2''$$

$$4I_1'+3I_2'=-6I_1''+8I_2''$$

Solving these equations we get

$$I_1'=80$$
$$I_2'=-40$$
$$I_1''=20$$
$$I_2''=40$$

Whence the current in No. 1 from a to b is

$$\dot{I}'_{ab}=80-j40$$

which has the effective value

$$I'=\sqrt{(80)^2+(40)^2}=89.4 \text{ amperes}$$

and lags behind the total current $\dot{I}_{ab}$ by the angle

$$tan^{-1}\frac{40}{80}=26.5°$$

The current in No. 2 from a to b is

$$I''_{ab}=20+j40$$

which has the effective value

$$I''=\sqrt{(20)^2+(40)^2}=44.7 \text{ amperes}$$

and leads the total current $\dot{I}_{ab}$ by the angle

$$tan^{-1}\frac{40}{20}=63.5°$$

The potential drop across each impedance in the direction from a to b is

$$\dot{V}_{ab}=(3+j4)\ (80-j40)=240+160+j\ (320-120)$$
$$=400+j200$$

which has the effective value

$$V=\sqrt{(400)^2+(200)^2}=447 \text{ volts}$$

and leads the total current by the angle

$$tan^{-1}\frac{200}{400}=26.5$$

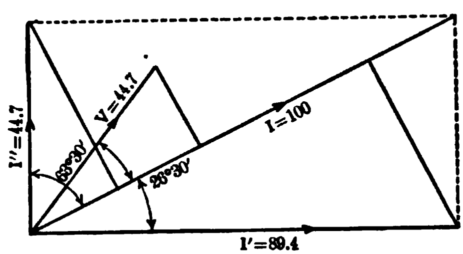


Fig. 125.

(Note, this problem can be solved more simply by calculating

the admittances corresponding to the two impedances by equations (37) of Chapter VII, and then proceeding in an entirely analogous manner to that employed in Problem 1. The method above given, however, illustrates the manner in which any problem concerning a network of circuits may be attacked.)

The vector diagram for the currents and *p.d.* is given in Fig. 125.

The *p.d.* is plotted to one-tenth the scale of the current.

The power input into No. 1 is

$$W' = 80 \times 400 - 40 \times 200 = 24{,}000 \text{ watts}$$

The power input into No. 2 is

$$W'' = 20 \times 400 + 40 \times 200 = 16{,}000 \text{ watts}$$

The total power input is

$$W = 100 \times 400 + 0 \times 200 = 40{,}000 \text{ watts}$$

SUMMARY OF IMPORTANT DEFINITIONS AND PRINCIPLES

1. A **vector** of length A making an angle θ with any arbitrarily chosen axis of reference may be **represented symbolically** by the expression

$$\underset{\cdot}{A} = A_1 + jA_2$$

where A_1 is the component $(= A \cos \theta)$ of the vector A along the axis and A_2 is the component $(= A \sin \theta)$ of the vector A making a positive angle of 90° (counter-clockwise) with the axis. The symbol "j" indicates that the component A_2 makes the angle 90° with the axis.

2. The **length of any vector** $A_1 + jA_2$ is $A = \sqrt{A_1^2 + A_2^2}$ and the **angle** which it makes **with the axis** is $\theta = \tan^{-1} \dfrac{A_2}{A_1}$.

3. The **sum of two vectors** $A_1 + jA_2$ and $A_1' + jA_2'$ referred to the same axis is

$$\underset{\cdot}{S} = (A_1 + A_1') + j(A_2 + A_2')$$

4. In addition to showing direction the **symbol** j **also has the following properties**:

(*a*) Multiplying a vector by j rotates the vector through 90° in the positive direction.

(*b*) $jj = j^2 = -1$.

(*c*) $jK = Kj$ when K is a constant, real or complex.

(*e*) $j\underset{\cdot}{\dot{X}} \pm \underset{\cdot}{\dot{X}}j$ when $\underset{\cdot}{\dot{X}}$ is a rotating vector.

(*d*) $j\underset{\cdot}{X} = \frac{1}{\omega} \frac{d\underset{\cdot}{X}}{dt}$ when $\underset{\cdot}{X}$ is a vector rotating with a constant angular velocity ω.

5. The **symbolic notations for a sine-wave current, a sine-wave** *p.d.*, **and a sine-wave** *e. m. f.* are respectively

$$\underset{\cdot}{I} = I_1 + jI_2$$
$$\underset{\cdot}{V} = V_1 + jV_2$$
$$\underset{\cdot}{E} = E_1 + jE_2$$

where the two components of each quantity are any two components at right angles to each other and each component is a vector rotating with an angular velocity $\omega = 2\pi f$, where f is the frequency.

6. The **symbolic notation for an impedance** of constant resistance r and constant reactance x to a sine-wave current is the complex number

$$\underset{\cdot}{Z} = r + jx$$

7. The **symbolic notation for an admittance** of constant conductance g and constant susceptance b to a sine-wave current is the complex number

$$y = g - jb$$

8. The symbolic notation for the **p.d.** $\underset{\cdot}{\mathbf{V}}$ **in an impedance** $\underset{\cdot}{\mathbf{z}}$ due to a current $\underset{\cdot}{I}$ is

$$\underset{\cdot}{V} = \underset{\cdot}{z}\underset{\cdot}{I}$$

9. The symbolic notation for the **current** $\underset{\cdot}{\mathbf{I}}$ **in an admittance** $\underset{\cdot}{\mathbf{y}}$ due to a *p.d.* $\underset{\cdot}{V}$ is

$$\underset{\cdot}{I} = \underset{\cdot}{y}\underset{\cdot}{V}$$

10. The **relation between impedance and admittance** in symbolic notation is

$$\underset{\cdot}{y} = \frac{1}{z}$$

11. The symbolic notation for the equivalent impedance Z of two **impedances** $\underset{\cdot}{z}_1$ **and** $\underset{\cdot}{z}_2$ **in parallel** is

$$Z = \frac{\underset{\cdot}{z}_1 + \underset{\cdot}{z}_2}{\underset{\cdot}{z}_1 \underset{\cdot}{z}_2}$$

12. An expression of the form $\frac{1}{a_1 \pm ja_2}$ where a_1 and a_2 are constants is equal to

$$\frac{a_1 \mp j\, a_2}{a_1^2 + a_2^2}$$

13. **Kirchhoff's Laws** in symbolic notation are

$$\Sigma\, \dot{I} = 0$$

$$\Sigma\, \dot{z}\, \dot{I} = \Sigma\, \dot{E}$$

where the currents $\dot{I}$ and the electromotive forces $\dot{E}$ are all referred to the same axis of reference.

14. The **average power** corresponding to a *p.d.* $\dot{V} = V_1 + j\, V_2$ and a current $\dot{I} = I_1 + jI_2$, both referred to the same axis of reference is

$$P = V_1\, I_1 + V_2\, I_2$$

and the **power factor** is

$$\cos\theta = \frac{V_1\, I_1 + V_2\, I_2}{V\, I}$$

PROBLEMS

1. A vector A 10 units in length makes an angle of 60° in a counter-clockwise direction with another vector B 15 units in length. Find the symbolic expressions of (1) A and B respectively, and (2) $\overline{A+B}$, and (3) $\overline{A-B}$ referred to B as the axis of reference in each case.

Ans.: (1) $5 + j\,8.66$ and 10; (2) $20 + j\,8.66$; (3) $-10 + j\,8.66$.

2. A certain vector is represented symbolically by the expression $12 + j24$. Find the symbolic expression of a vector equal in length (1) when it makes an angle of $-40°$ with the axis of reference, and (2) when it makes a right angle with the axis of reference in the clockwise direction.

Ans.: (1) $20.6 - j17.27$; (2) $-j\,26.8$.

3. Three impedances A, B and C have resistances of 5, 8 and 3 ohms respectively and reactances of 14, 0 and -10 ohms respectively. Find the symbolic expressions of the resultant impedance of these three impedances, (1) when they are connected in series, (2) when they are connected in parallel, and (3) find the symbolic expression of the resultant admittance of these three impedances when they are connected in parallel.

Ans.: (1) $16 + j4$; (2) $5.56 - j0.902$; (3) $0.1751 + j0.0284$.

4. An *e. m. f.* of 150 volts and 25 cycles is impressed upon a series circuit consisting of a resistance of 5 ohms, an inductance of 0.05 henry and a capacity of 1000 microfarads. If the vector

representing the impressed *e. m. f.* is taken as the axis of reference find (1) the symbolic expression for the current flowing in the circuit, and (2) the average rate at which energy is given to the circuit.

Ans.: (1) $27.6 - j\,8.17$; (2) 4140 watts.

5. Two impedances A and B of 4 and 12 ohms resistance respectively and 15 and -8 ohms reactance respectively are connected in series. If the current established in this circuit is 10 amperes, find the symbolic expression for the potential drop, (1) through A, (2) through B, and (3) through the entire circuit, all referred to the current as the axis of reference.

Ans.: (1) $40 + j150$; (2) $120 - j80$; (3) $160 + j70$.

6. If the two impedances described in Problem 5 are connected in parallel, and the current in A is 5 amperes, find the symbolic expressions for the currents in A and B respectively referred to the potential drop across the parallel circuit as the axis of reference.

Ans.: $1.289 - j4.84$ and $4.48 + j2.98$.

7. When a given 60-cycle *e. m. f.* is impressed upon a series circuit of 5 ohms resistance, 0.1 henry inductance and 50 microfarads capacity, the symbolic expression for the fundamental of the current is $12 - j10$ and for the third harmonic of the current $6 + j8$ referred respectively to the first and third harmonics respectively of the impressed *e. m. f.* Find (1) the effective value of the impressed *e. m. f.* and (2) the average power dissipated as heat energy in the circuit.

Ans.: (1) 988 volts; (2) 1720 watts.

8. An impedance A is connected in series with two impedances B and C connected in parallel. The resistances of A, B and C are 3, 5 and 6 ohms respectively and the reactances are 5, -7 and 3 ohms respectively. If an *e. m. f.* of 200 volts is impressed across the entire circuit, find the symbolic expression of the currents in A, B and C referred to the potential drop in A.

Ans.: $11.4 - j19.0$; $12.4 - j2.75$; $-1.01 - j16.3$.

9. An *e. m. f.* of 100 volts is impressed upon a series circuit formed by two impedances A and B in series. The impedance A has a resistance of 4 ohms and a reactance of 6 ohms and the potential drop across A is 100 volts. If the entire circuit absorbs 1 kilowatt of power, find the symbolic expression of (1) the current, (2) the potential drop through A, and (3) the potential drop through B, all referred to the *e. m. f.* impressed upon the entire circuit.

Ans.: (1) $10 \pm j9.58.$; (2) $-17.5+j98.3$ or $97.5+j21.7$; (3) $117.5 -j98.3$ or $2.5-j21.7$.

10. Two impedances A and B are connected in parallel. The respective symbolic expressions for the impedances of A and B are $4+j8$ and $5-j3$. If the total current supplied to this parallel circuit is 20 amperes, find the symbolic expressions for (1) the currents in A and B, and (2) the potential drop across the parallel circuit, all referred to the total current supplied to the circuit.

Ans.: (1) $5.66-j9.81$ and $14.34+j9.81$; (2) $101.1+j6.0$.

11. A 500-volt, 60-cycle alternator is delivering 25 kw. at a power factor of 85%, lagging current. The armature has a resistance of 0.25 ohm and an inductance of 0.001 henry. Referring all vectors to the terminal voltage of the alternator, find the symbolic expression for (1) the armature current, (2) the armature voltage, and (3) the resistance drop in the armature.

Ans.: (1) $50.0-j31.0$; (2) $524+j11.1$; (3) $12.5-j7.75$.

12. The generator in Problem 11 supplies power to a motor over a line of 0.3 ohm resistance and $+0.6$ ohm reactance. If the armature of the motor has a resistance of 0.2 ohm and a reactance of 0.3 ohm, find the symbolic expression of the armature voltage of the motor referred to the terminal voltage of the motor.

Ans.: $-448.1+j9.7$.

IX

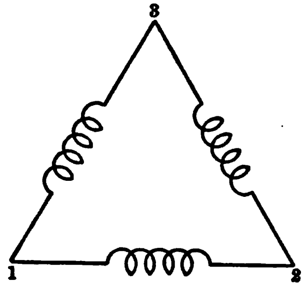
THREE-PHASE ALTERNATING CURRENTS

219. Polyphase Alternating Currents. — A polyphase alternator is an alternator upon the armature core of which are wound two or more independent windings which are arranged with respect to each other in such a manner that the electromotive forces in the various windings differ in phase by a constant angle. In a two-phase alternator there are two independent windings, which are arranged in such a manner that when the *e. m. f.* induced in one winding is a maximum, the *e. m. f.* induced in the other winding is zero, that is, the *e. m. f.* induced in one winding is in quadrature with the *e. m. f.* induced in the other winding; this effect is obtained by placing the two windings on the core in such a manner that when a group of conductors in one winding is directly under a north pole, the corresponding group of conductors in the second winding is midway between a north pole and a south pole. In the case of a two-pole machine this means that a group of conductors in one winding is placed 90 degrees ahead of the corresponding group in the other winding; similarly, in the case of a multipolar machine, if we call *the distance between two successive poles of like sign equal to* 360 *"electrical" degrees,* we may describe the relative position of the two windings as such that the corresponding conductors in the two windings are spaced 90 *electrical degrees* apart. In general the two windings overlap each other, or are "distributed" over the armature surface, but for each conductor in one winding there is a corresponding conductor in the other winding 90 electrical degrees, or one quarter of the pole pitch, ahead of the conductor in the first winding.

A two-phase alternator may be provided with a separate pair of slip rings for each winding, that is, four rings in all; or with three slip rings, and one of these rings made to serve as a common return for the two windings.

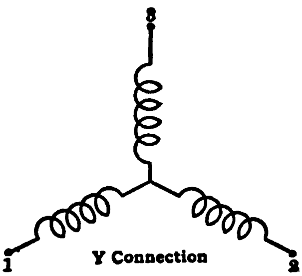
A three-phase alternator is similar in construction to a two-phase machine except that three separate windings are employed, the corresponding conductors in the three windings being spaced

120 *electrical degrees* apart, and the induced electromotive forces therefore differ in phase by 120°. These windings are connected electrically in two different ways. The three windings may be connected end to end as shown in Fig. 126, and leads from the junctions 1, 2 and 3, brought out to three slip rings; or one end of each of the three windings may be connected to a common junction, called the "neutral point," as shown in Fig. 127, and the other ends 1, 2 and 3 brought out to three slip rings. In some cases the common junction or neutral point is also connected to a fourth slip ring. A three-phase alternator with its windings connected as shown in Fig. 126 is said to be "mesh" or "Delta" connected, the latter from the Greek capital letter "Δ." When the windings are connected as shown in Fig. 128 the alternator is said to be "star" or "Y" connected. In large generators the armature is stationary and the field rotates; in such a case the armature windings are connected to fixed terminals and the two ends of the field winding are connected to two slip rings. Continuous current is supplied to the field through brushes bearing on these two slip rings.



Δ Connection

Fig. 126.



Y Connection

Fig. 127.

The advantages in using three-phase currents are chiefly:

1. The more satisfactory operation of three-phase motors as compared with single-phase machines.

2. The saving in the amount of copper (25 per cent) as compared with a single-phase system for the same voltage between wires.

3. The lesser cost of three-phase generators and motors of the same power and for the same voltage between terminals.

4. Better voltage regulation of three-phase generators.

220. Vector Sum of the Induced E. M. F.'s in the Three Windings of a Three-Phase Generator Equals Zero. — Let $\dot{E}_{12}$, $\dot{E}_{23}$ and $\dot{E}_{31}$, Fig. 128, be the three *e.m.f.'s* induced in the three wind-

ings of a three-phase generator, *taken as positive in the counter-clockwise direction around the closed loop formed by the windings of*

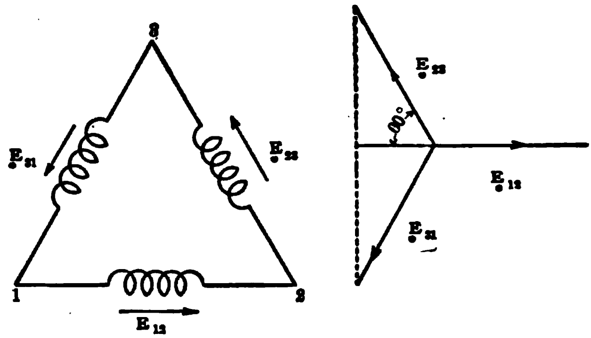


Fig. 128.

the generator. In a properly designed machine these *e.m.f.'s* have equal effective values, but due to the relative position of the windings they differ in phase by 120°; that is $\dot{E}_{12}$ leads $\dot{E}_{23}$ by 120° and $\dot{E}_{31}$ leads $\dot{E}_{12}$ by 120°. Hence the symbolic expressions for the three *e.m.f.'s* referred to $\dot{E}_{12}$ as the line of reference are, when these *e.m.f.'s*. are sine waves,

$$\begin{aligned} \dot{E}_{12} &= (\ 1 + j\ 0\ )\, E_{12} \\ \dot{E}_{23} &= \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) \dot{E}_{12} \\ \dot{E}_{31} &= \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) \dot{E}_{12} \end{aligned} \qquad (1)$$

and therefore

$$\dot{E}_{12} + \dot{E}_{23} + \dot{E}_{31} = 0$$

Hence the total *e.m.f.* at any instant acting around the closed loop formed by the three windings is zero, and therefore when no external circuit is connected to the three terminals, *i.e.*, when the generator is running on " open " (external) circuit, there will be no flow of current in the armature provided the *e.m.f.'s* are sine waves.

Similarly, in the case of a three-phase *Y*-connected generator, Fig. 129, the induced *e.m.f.'s* $\dot{E}_{01}$, $\dot{E}_{02}$, and $\dot{E}_{03}$ are all equal in effective value and differ in phase by 120°. Hence choosing $\dot{E}_{01}$ as the line of reference and *taking these e.m.f.'s as positive in the*

direction away from the neutral point, we have, when the *e.m.f.*'s are sine waves, that

$$\begin{aligned}\dot{E}_{01} &= \left(\ 1 + j\,0\ \right)\dot{E}_{01}\\ \dot{E}_{02} &= \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\dot{E}_{01}\\ \dot{E}_{03} &= \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\dot{E}_{01}\end{aligned} \qquad (2)$$

and therefore

$$\dot{E}_{01} + \dot{E}_{02} + \dot{E}_{03} = 0$$

When the generator is running on open (external) circuit there

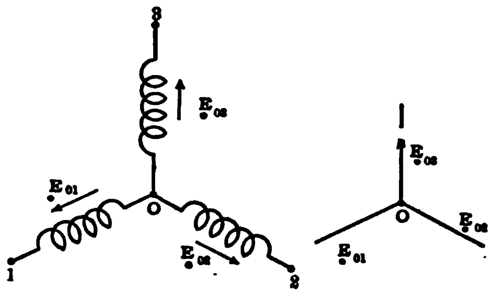


Fig. 129.

is of course no current in the coils, since each coil is open at one end.

221. Relation Between Coil E.M.F. and E.M.F. Between Terminals in a Y-Connected Three-Phase Generator. —The *e.m.f.* between any pair of terminals, or the equivalent "Δ" *e.m.f.* of a *Y*-connected generator is equal to the vector sum of the *e.m.f.'s* in the two coils *in the direction from one terminal to the other.* Using the above notation, and assuming sine-wave *e.m.f.'s,* we have that the *e.m.f.* between the terminals 2 and 3 in the *Y*-connected generator in the direction from 2 to 3 is

$$\begin{aligned}\dot{E}_{23} &= \dot{E}_{20} + \dot{E}_{03} = -\dot{E}_{02} + \dot{E}_{03}\\ &= -\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\dot{E}_{01} + \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\dot{E}_{01}\\ &= -j\sqrt{3}\,\dot{E}_{01}\end{aligned}$$

Similarly for the *e.m.f.*'s between the terminals 3 and 1 and

between the terminals 1 and 2; hence, the equivalent "Δ" *e. m. f.'s*, that is, the *e.m.f.'s* between terminals, are

$$\dot{E}_{23} = -j\sqrt{3}\,\dot{E}_{01}$$
$$\dot{E}_{31} = -j\sqrt{3}\,\dot{E}_{02} \qquad (3)$$
$$\dot{E}_{12} = -j\sqrt{3}\,\dot{E}_{03}$$

That is, the equivalent Δ *e.m.f.* is equal numerically to the square root of three times the effective value of the *Y e. m. f.* and lags 90° behind the *e.m.f.* in the opposite branch of the *Y*. The Δ *e.m.f.'s* are therefore represented by the sides of the triangle formed by connecting the ends of the three vectors representing the *Y e. m. f.'s*, the positive direction being taken in the counter-clockwise direction, see Fig. 130.

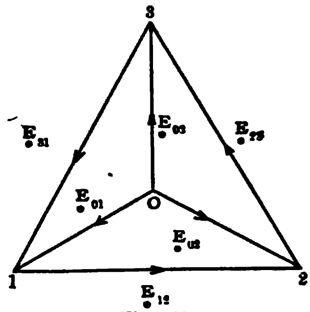


Fig. 130.

222. Currents from a Three-Phase Generator. — Generator and Load Both Y-Connected. —When the three terminals of a three-phase generator, either Δ or *Y* connected, are connected to either a *Y* or Δ connected load, currents equal in effective value and differing in phase by 120° will flow from each terminal, provided (1) that the resistance and reactances of the generator windings are respectively the same for the three windings, (2) that the resistances and reactances of the windings of the receiving device or load are respectively the same for the three windings, (3) that the *e. m. f.'s* in each winding of the generator are equal in effective value and differ in phase by 120°, and (4) that the back *e. m. f.'s*, if any, in the windings of the load are equal in effective value and differ in phase by 120°. When the *e. m. f.'s* in each phase of a three-phase system are equal in effective value and differ in phase by 120° and the currents in each phase are equal in effective value and differ in phase by 120°, the system is said to be "balanced."

Consider the simple case of a *Y*-connected generator to the three terminals of which are connected three equal impedances in *Y*, Fig. 131. Let $\dot{z}$ represent the total impedance of each winding of the load plus, *vectorially*, the impedance of the generator winding in series with it. For the closed loops $01a0'b20$ and $02b0'c30$, we have, assuming sine-wave currents and *e.m.f.'s*, that

$$\underset{\cdot}{E}_{01} - \underset{\cdot}{E}_{02} = \underset{\cdot}{z}\underset{\cdot}{I}_{01} - \underset{\cdot}{z}\underset{\cdot}{I}_{02} \qquad (a)$$

$$\underset{\cdot}{E}_{02} - \underset{\cdot}{E}_{03} = \underset{\cdot}{z}\underset{\cdot}{I}_{02} - \underset{\cdot}{z}\underset{\cdot}{I}_{03} \qquad (b)$$

Also at the junction 0 or 0′ we have that

$$\underset{\cdot}{I}_{01} + \underset{\cdot}{I}_{02} + \underset{\cdot}{I}_{03} = 0 \qquad (c)$$

which give three equations in the three unknown quantities $\underset{\cdot}{I}_{01}$ $\underset{\cdot}{I}_{02}$ and $\underset{\cdot}{I}_{03}$. Referring the *e.m.f.'s* to $\underset{\cdot}{E}_{01}$ as the axis of reference, equations (*a*) and (*b*) may be written

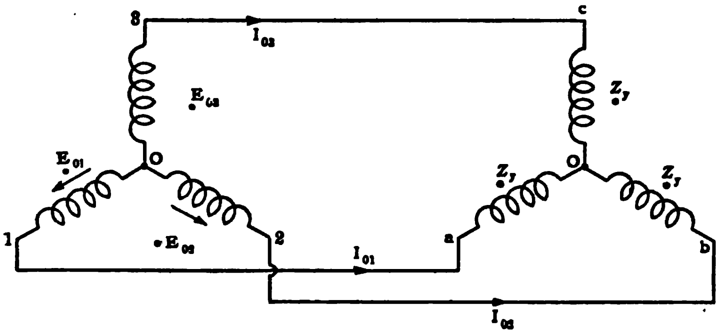


Fig. 131.

$$(1+j0)\ \underset{\cdot}{E}_{01} - \left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right) \underset{\cdot}{E}_{01} = \left(\frac{3}{2}-j\frac{\sqrt{3}}{2}\right) \underset{\cdot}{E}_{01} = \underset{\cdot}{z}\left(\underset{\cdot}{I}_{01} - \underset{\cdot}{I}_{02}\right) \quad (d)$$

$$\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right) \underset{\cdot}{E}_{01} - \left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\underset{\cdot}{E}_{01} = j\sqrt{3}\ \underset{\cdot}{E}_{01} = \underset{\cdot}{z}\left(\underset{\cdot}{I}_{02} - \underset{\cdot}{I}_{03}\right) \quad (e)$$

Substituting the value of $\underset{\cdot}{I}_{03}$ from equation (*c*) in the second of these equations, gives

$$j\sqrt{3}\ \underset{\cdot}{E}_{01} = \underset{\cdot}{z}\ (2\ \underset{\cdot}{I}_{02} + \underset{\cdot}{I}_{01}) \qquad (f)$$

which subtracted from (*d*) gives

$$\left(\frac{3}{2}-j\frac{3\sqrt{3}}{2}\right) \underset{\cdot}{E}_{01} = -3\ \underset{\cdot}{z}\underset{\cdot}{I}_{02}$$

Therefore

$$\underset{\cdot}{z}\underset{\cdot}{I}_{02} = \left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right) \underset{\cdot}{E}_{01} = \underset{\cdot}{E}_{02}$$

Similarly for the other two phases; hence

$$\begin{aligned} \underset{\cdot}{z}\underset{\cdot}{I}_{01} &= \underset{\cdot}{E}_{01} \\ \underset{\cdot}{z}\underset{\cdot}{I}_{02} &= \underset{\cdot}{E}_{02} \\ \underset{\cdot}{z}\underset{\cdot}{I}_{03} &= \underset{\cdot}{E}_{03} \end{aligned} \qquad (4)$$

Consequently, since the three *e.m.f.'s* are equal in effective value and differ in phase by 120°, the currents in the three impedances have equal effective values and differ in phase by 120°. These currents, however, are *not in phase with the corresponding e.m.f.'s* except in the special case when the reactance component of the impedance is zero.

When the load is a motor or other device developing in the three windings back *e.m.f.'s* equal in effective value and differing in phase by 120°, it can readily be shown that equations (4) still hold *provided* the *E's* in these equations are taken to represent the vector difference of the generator *e.m.f.* per winding and the back *e.m.f.* of the load per winding.

From the above discussion it therefore follows that when a balanced *Y*-connected load is connected to a *Y*-connected generator, the current established in each winding of the load is exactly the same as would be produced were this winding connected by itself to a single-phase generator having an *e. m. f.* equal to the *e. m. f.* induced in each winding of the generator and an internal impedance equal to the impedance of each of these windings. It also follows from equation (4) that the neutral point of the generator and the neutral point of the load must be at the same potential, since the rise of potential in the generator from the neutral point to any one slip ring due to the induced *e. m. f.* is equal to the drop in potential $\dot{z}\dot{I}$ due to the resultant impedance of the generator, line and load. Consequently, were the neutral points of the generator and load connected by a wire, no current would flow in this "neutral" wire, *provided the system is balanced.* In case the load is not balanced, a current would flow in the neutral wire; such a wire is sometimes installed to reduce the voltage drop between generator and load in case of unbalancing. In general, however, a three-phase system is very nearly balanced, as three-phase motors and rotary converters are designed to take the same current per phase, and in the case of a lamp load or single-phase motors used on a three-phase system, care is taken to distribute the various lamps and single-phase motors equally on the three-phases.

223. Currents from a Three-Phase Generator.—Generator Δ-Connected and Load Y-Connected. — When the generator is Δ-connected the induced *e.m.f.'s* between terminals are $\dot{E}_{12}$, $\dot{E}_{23}$ and $\dot{E}_{31}$. Hence, neglecting the resistance and reactance of the generator, and assuming sine-wave *e.m.f.'s* and currents, we have

$$\dot{E}_{12} = z\,(\dot{I}_{03} - \dot{I}_{01}) \qquad (a)$$
$$\dot{E}_{23} = z\,(\dot{I}_{03} - \dot{I}_{02}) \qquad (b)$$
$$\dot{I}_{01} + \dot{I}_{02} + \dot{I}_{03} = 0 \qquad (c)$$

Referring $\dot{E}_{12}$ and $\dot{E}_{23}$ to $\dot{E}_{12}$ as the line of reference and substituting in (a) and (b) the value of $\dot{I}_{02}$ from equation (c), we have

$$(1 + j0)\ \dot{E}_{12} = -z\,(\dot{I}_{03} + 2\dot{I}_{01}) \qquad (d)$$
$$\left(-\frac{1}{2} + j\,\frac{\sqrt{3}}{2}\right)\dot{E}_{12} = z\,(2\dot{I}_{03} + \dot{I}_{01}) \qquad (e)$$

Multiplying the second equation by 2 and adding it to the first gives

$$j\sqrt{3}\ \dot{E}_{12} = 3\ \dot{z}\dot{I}_{03}$$

or

$$\dot{z}\dot{I}_{03} = \frac{j\dot{E}_{12}}{\sqrt{3}}$$

Similarly for the other two phases; whence

$$\left.\begin{aligned} \dot{z}\dot{I}_{03} &= \frac{j\dot{E}_{12}}{\sqrt{3}} \\ \dot{z}\dot{I}_{01} &= \frac{j\dot{E}_{23}}{\sqrt{3}} \\ \dot{z}\dot{I}_{02} &= \frac{j\dot{E}_{31}}{\sqrt{3}} \end{aligned}\right\} \qquad (5)$$

Hence, since the three Δ *e.m.f.'s* are equal in effective value and differ in phase by 120°, the currents in the three impedances are equal in effective value and differ 120° in phase. Note that the vectors $\frac{j\dot{E}_{12}}{\sqrt{3}}$, $\frac{j\dot{E}_{23}}{\sqrt{3}}$ and $\frac{j\dot{E}_{31}}{\sqrt{3}}$ are equal to the vectors represented by the lines drawn from the center of the triangle formed by the vectors $\dot{E}_{12}$, $\dot{E}_{23}$ and $\dot{E}_{31}$; hence the currents taken from slip rings of a Δ-connected generator by a balanced load are, *neglecting armature impedance,* exactly the same as would be produced by a Y-connected generator developing *e.m.f.'s* $\dot{E}_{01}$, $\dot{E}_{02}$, $\dot{E}_{03}$ per winding where

$$\left.\begin{aligned} \dot{E}_{12} &= -j\sqrt{3}\ \dot{E}_{03} \\ \dot{E}_{23} &= -j\sqrt{3}\ \dot{E}_{01} \\ \dot{E}_{31} &= -j\sqrt{3}\ \dot{E}_{02} \end{aligned}\right\} \qquad (5a)$$

Compare with equations (3). *A Δ-connected generator, when sup-*

plying a balanced load is therefore equivalent to a Y-connected generator developing an e.m.f. per winding E_y numerically equal to $\frac{E_\Delta}{\sqrt{3}}$, where E_Δ is the e.m.f. per winding of the Δ-connected generator. As will be shown presently, when the impedance of the generator is taken into account, as it must be in all practical problems, it is also necessary, in order that the Y-connected and the Δ-connected generator produce the same current in the external circuit, that the impedance per winding of the equivalent Y-connected generator be equal to *one-third* the impedance per winding of the Δ-connected generator.

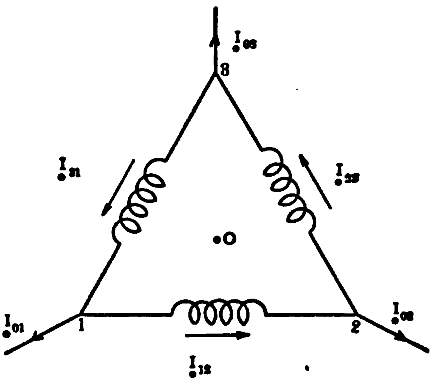


Fig. 132.

224. Coil Current in a Δ-Connected Generator for Balanced Load. — In a Δ-connected generator the current I taken from each terminal of the generator must be equal to the vector sum of the currents flowing *to* that terminal in the windings of the generator. Let I_{12}, I_{23} and I_{31} be the currents in the three windings in the directions indicated in Fig. 133, and as before let I_{01}, I_{02} and I_{03} be the currents taken from the three terminals. The first set of currents are called the " Δ " or coil currents and the second set the " Y " or line currents. *When the load is balanced,* the three Y currents are equal in effective value and differ in phase by 120°, see Article 223. Hence, referring all the currents to I_{03} as the axis of reference, and assuming sine-wave currents and *e. m. f.'s*, we have that

$$I_{23} - I_{31} = (\ 1 + j\,o\ )\, I_{03} \qquad (a)$$

$$I_{31} - I_{12} = \left(-\frac{1}{2} + j\,\frac{\sqrt{3}}{2}\right) I_{03} \qquad (b)$$

$$I_{12} + I_{23} + I_{31} = 0 \qquad (c)$$

Hence, substituting in (a) and (b) the value of I_{31} from (c) we have

$$2\, I_{23} + I_{12} = (1 + j0)\, I_{03} \qquad (d)$$

and
$$-(2\,\underset{\cdot}{I}_{12}+\underset{\cdot}{I}_{23})=\left(-\frac{1}{2}+j\,\frac{\sqrt{3}}{2}\right)\underset{\cdot}{I}_{03} \qquad (e)$$

Multiplying (e) by 2 and adding (d) gives
$$-3\,\underset{\cdot}{I}_{12}=j\,\sqrt{3}\,\underset{\cdot}{I}_{03}$$

whence
$$\underset{\cdot}{I}_{12}=-\frac{\dot{j}}{\sqrt{3}}\,\underset{\cdot}{I}_{03}$$

Similarly for the other two phases; whence
$$\begin{aligned}\underset{\cdot}{I}_{12}&=-\frac{\dot{j}}{\sqrt{3}}\,\underset{\cdot}{I}_{03}\\ \underset{\cdot}{I}_{23}&=-\frac{\dot{j}}{\sqrt{3}}\,\underset{\cdot}{I}_{01}\\ \underset{\cdot}{I}_{31}&=-\frac{\dot{j}}{\sqrt{3}}\,\underset{\cdot}{I}_{02}\end{aligned} \qquad (6)$$

Hence the current in each of the windings of a Δ-connected generator is equal numerically to the effective value of the Y or line current divided by the square root of three, and lags 90° behind the current in the opposite branch of the Y, provided the system is perfectly balanced. Compare with the relation between the Δ *e. m. f.'s* and the Y *e. m. f.'s* given by equation (3).

The Y currents, like the Y *e. m. f.'s*, may be represented by three equal vectors differing in phase by 120°. The sides of the triangle formed by joining the ends of these vectors are then equal to *three times* the currents, but their directions, taken positive in the counter-clockwise direction, give the proper phase relation between the Δ and Y currents, see Fig. 133.

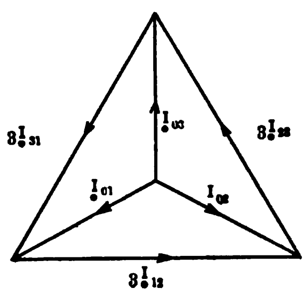


Fig. 133.

225. Coil E. M. F. and Impedance of the Equivalent Y-Connected Generator and of the Equivalent Y-Connected Load.— A Y-connected and a Δ-connected generator may be looked upon as equivalent to each other provided the voltage between terminals is the same in each for the same current taken from the terminals. Let E_Δ, I_Δ and z_Δ, and E_y, I_y and z_y, be the effective *e.m.f.* per winding, the effective

current per winding and impedance per winding for a Δ and Y connected generator respectively. On the assumption of sine-wave currents and *e. m. f.*'s, the voltage between terminals and 2 in case of the Δ-connected generator is the vector difference $\dot{E}_{12} - \dot{z}_\Delta \dot{I}_{12}$, and the voltage between terminals in the case of the Y-connected generator is, when referred to $\dot{E}_{12}$ as the line of reference, the vector difference $-j\sqrt{3}\,(\dot{E}_{03} - \dot{z}_y \dot{I}_{03})$, provided the system is balanced, see equations (3). Hence in order that these two expressions be equal for all values of the line or Y current we must have

$$\dot{E}_{12} = -j\sqrt{3}\,\dot{E}_{03}$$

and

$$\dot{z}_\Delta \dot{I}_{12} = -j\sqrt{3}\,\dot{z}_y \dot{I}_{03}$$

But from equation (6) $\dot{I}_{12} = -\dfrac{j}{\sqrt{3}}\dot{I}_{03}$; therefore

$$\dot{z}_\Delta = 3\dot{z}_y \tag{7}$$

Therefore, in order that the voltage between terminals be the same in a Y-connected as in a Δ-connected generator, the coil *e. m. f.* in the Y-connected generator must be equal to $\dfrac{1}{\sqrt{3}}$ times the coil *e. m. f.* of the Δ-connected generator and both the resistance and the reactance of each winding of the Y-connected generator must be one-third the resistance and reactance respectively of the Δ-connected generator.

By a similar argument it can be shown that a Δ-connected load may be considered equivalent to a Y-connected load provided the back *e. m. f.* in each winding of the equivalent Y is taken equal to $\dfrac{1}{\sqrt{3}}$ times the coil *e. m. f.* of each winding of the Δ-connected load and both the resistance and reactance of each winding of the equivalent Y are taken equal to one-third the resistance and reactance respectively of each winding of the Δ.

226. Reduction of all Balanced Three-Phase Circuits to Equivalent Y's. — Any problem in regard to a *balanced* three-phase circuit may therefore be solved by reducing all parts of the circuit to an equivalent Y connection, provided the currents and *e. m. f.*'s are sine waves. The transformations are made as follows:

Any Δ-connected motor or generator is considered as equiv-

alent to a Y-connected generator or motor in which the *e. m. f.* per winding E_y is equal to $\frac{1}{\sqrt{3}}$ times the *e. m. f.* per winding E_Δ of the Δ-connected machine, *i.e.*,

$$E_y = \frac{E_\Delta}{\sqrt{3}} \tag{8a}$$

and the impedance per winding $\dot{z}_y$ of the equivalent Y-connected machine as equal to the impedance per winding $\dot{z}_\Delta$ of the Δ-connected machine divided by three, *i.e.*,

$$\dot{z}_y = \frac{\dot{z}_\Delta}{3} \tag{8b}$$

Similarly, any Δ-connected load which has an impedance per winding equal to $\dot{z}_\Delta$ may be replaced by a Y-connected load which has an impedance per winding

$$\dot{z}_y = \frac{\dot{z}_\Delta}{3} \tag{8c}$$

The current per winding I_Δ in any Δ-connected generator or load is equivalent to a current I_y in the equivalent Y-connected generator or load, where

$$I_y = \sqrt{3}\, I_\Delta \tag{8d}$$

or *vice versa*, when the current in the equivalent Y is found to be I_y, the corresponding current per winding in the Δ is

$$I_\Delta = \frac{I_y}{\sqrt{3}} \tag{8e}$$

Similarly, when the equivalent Y voltage is found to be E_y, the corresponding Δ voltage is

$$E_\Delta = \sqrt{3}\, E_y \tag{8f}$$

Each of the wires (*e.g.*, the wire $3c$ in Fig. 131) connecting a generator terminal to a load terminal is then in series with the corresponding phase of the equivalent Y generator and Y load.

When all parts of the circuit have thus been reduced to equivalent Y's, each of the three phases may be treated as a single-phase circuit, each circuit considered completed by a wire having *zero* impedance connecting all the neutrals together, since all the neutrals are at the same potential.

227. Power in a Balanced Three-Phase System. — Consider first a Y-connected load, and let the potential drop per winding be V_y, and the current per winding I_y, and the difference in

phase between current and *p.d. in this winding* be θ_y, then the power input into each winding is $V_y I_y \cos\theta_y$, and therefore the total power input is

$$P = 3V_y I_y \cos\theta_y$$

Similarly, in the case of a Δ-connected load, for which V_Δ is the potential drop per winding, I_Δ the current per winding, and θ_Δ the difference in phase between current and *p.d. in this winding*, the total power input is

$$P = 3V_\Delta I_\Delta \cos\theta_\Delta$$

In the case of a generator or any kind of a load, the *p.d.* that is most easily measured is the *p.d.* between terminals or the *p.d.* between the mains where they connect with the terminals. In case of a Δ-connected generator or load this is the *p.d.* per winding, that is, V_Δ; in the case of a Y-connected machine this is the equivalent Δ *p.d.* and is equal to the *p.d.* per winding V_y multiplied by the square root of three, that is

$$V_\Delta = \sqrt{3}\, V_y$$

Since V_Δ is the same as the *p.d.* between wires, V_Δ is frequently called the *line voltage.*

The current that is usually most readily measured is the current in each main, that is, the current leaving each terminal of a generator or entering each terminal of the load. In the case of a Y connection this is equal to the coil current I_y, and in the case of a Δ-connected machine it is the equivalent Y current and therefore is equal to the coil current I_Δ multiplied by the square root of three. That is,

$$I_y = \sqrt{3}\, I_\Delta$$

Since I_y is the same as the current per main, I_y is frequently called the *line current.* Hence in expressing the power input into a Y-connected load it is more convenient to substitute for V_y its value in terms of the line voltage V_Δ, which gives

$$P = 3\frac{V_\Delta}{\sqrt{3}} I_y \cos\theta_y$$

$$= \sqrt{3}\, V_\Delta I_y \cos\theta_y \qquad (9a)$$

In the case of a Δ-connected load, the substitution for I_Δ its value in terms of the line current I_y, gives

$$P = 3\, V_\Delta \frac{I_y}{\sqrt{3}} \cos\theta_\Delta$$

$$=\sqrt{3}\, V_{\Delta}\, I_y \cos \theta_{\Delta} \qquad (9b)$$

Equations (9) are the expressions usually employed for the three-phase power, and are usually written without the subscripts; namely, the total three-phase power is usually written

$$P=\sqrt{3}\; V\, I \cos \theta$$

It should be clearly borne in mind, however, that *the θ in this expression is not the phase angle between the voltage V and the current I,* but is the phase angle between the voltage to neutral and the line current, which in turn is equal to the phase angle between the voltage between wires and the Δ current.

228. Example. — Energy is supplied from a generating station to a substation 50 miles away at a rate of 20,000 kilowatts. The system is a balanced three-phase system and operates at a frequency of 25 cycles. The transmission line consists of three No. 0000 B. & S. copper wires spaced six feet between centers. It is desired to find (1) what will be the voltage between wires at the generating station when the voltage between wires at the substation is 60,000 volts, and the power factor at the substation is 80 per cent, with the current *lagging,* (2) how much power is lost in the transmission line, and (3) what is the power factor at the generating station. The electrostatic capacity of the line may be neglected.

The current per wire is

$$I=\frac{20,000,000}{\sqrt{3}\times 60,000\times 0.8}=241 \text{ amperes}$$

The voltage from line to neutral at the substation is

$$V_y=\frac{60,000}{\sqrt{3}}=34,600$$

Taking $\dot{I}_y$ as the line of reference, the symbolic notation for the voltage to neutral is

$$\dot{V}_y=34,600\ (\cos \theta+j \sin \theta)$$

where θ is the angle by which the voltage *leads* the line of reference I_y (see Article 207). But $\cos \theta=0.8$, and, since the current is *lagging* the angle θ by which the voltage leads the current is positive, and therefore $\sin \theta=+0.6$. Therefore

$$V_y=34,600\ (0.8+j\, 0.6)=27,700+j\, 20,800$$

The resistance per mile of a No. 0000 wire is 0.258 ohms; its inductance per mile for a spacing of six feet is 1.93 millihenries, hence the reactance per mile at 25 cycles of a No. 0000 wire is

$2\pi \times 25 \times 1.93 \times 10^{-3} = 0.303$ ohms. Hence the total impedance of each wire is

$$z_y = 50 \times 0.258 + j50 \times 0.303 = 12.9 + j15.2$$

Hence the voltage to neutral at the generating station is

$$V_y' = 27{,}700 + j20{,}800 + (12.9 + j15.2) \times 241$$
$$= 30{,}800 + j24{,}500$$

The effective value of which is 39,300 volts, and therefore the effective voltage between wires at the generating station is

$$V_\Delta' = \sqrt{3} \times 39{,}300 = 68{,}000$$

The power lost in the line is equal to $3RI_y^2$ where R is the total resistance of each wire and I_y the line current. Hence the power lost in the line is

$$3 \times 12.9 \times (241)^2 \text{ watts} = 2{,}250 \text{ kilowatts}$$

The total power delivered to the line and substation is then 22,250 kilowatts. Hence the power factor at the generating station is

$$\frac{22{,}250{,}000}{\sqrt{3} \times 68{,}000 \times 241} = 0.784 = 78.4\%$$

229. Rating of Three-Phase Apparatus. — The rated voltage of a three-phase machine always refers to the volts between terminals or the Δ voltage E_Δ, the power rating is the total power for all three-phases, and the power factor (usually written *cos* θ) is the ratio of the total power P to the square root of three times the voltage between terminals times the current per terminal I_y, and the machine is assumed to carry a balanced load. The current per terminal is then

$$I_y = \frac{P}{\sqrt{3}\, E_\Delta \cos\theta}$$

which is also the current per winding for a Y-connected machine. In the case of a Δ-connected machine the current per winding is

$$I_\Delta = \frac{P}{3\, E_\Delta \cos\theta}$$

230. Measurement of Power in a Three-Phase Circuit.—Two Wattmeter Method. — To measure the power output of a three-phase generator or the power input of a three-phase load, the most obvious method is to measure the power for each phase separately by connecting the current coil in the wattmeter in series with the winding of that particular phase and the potential coil of the wattmeter across the terminals of that winding. The

total power output (or input) is then the sum of the outputs (or inputs) for the three windings. In the case of a Y-connected load with an accessible neutral point this is readily accomplished, Fig. 134, by inserting the current coil in series with the line and connecting the potential coil between the terminal of load and its neutral point. (The output per phase of a Y-connected generator is similarly measured by connecting the current coil in series with that phase of the generator and the potential coil across the terminal of this phase and the neutral point of the generator.) In case the load is balanced, the reading of the wattmeter multiplied by three gives the total power. When the load is not balanced a similar measurement must be made for each phase and the three readings added; the three readings should be made simultaneously if the load is varying.

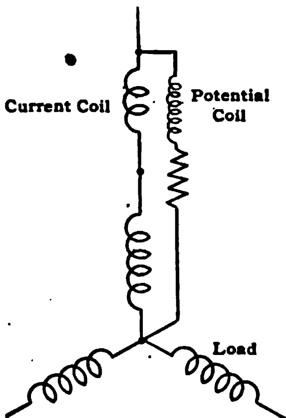


Fig. 134.

In the case of a Δ-connected generator or motor, the three windings are usually connected inside the machine, and it is therefore not feasible to connect the current coil in series with the windings. Again, the neutral point of a Y-connected generator or motor is frequently not accessible. The following method for measuring the power is then usually employed, and *gives the true power whether the system is balanced or not.* Two wattmeters A and B are connected as shown in Fig. 135; the current coil of A is connected in series with one

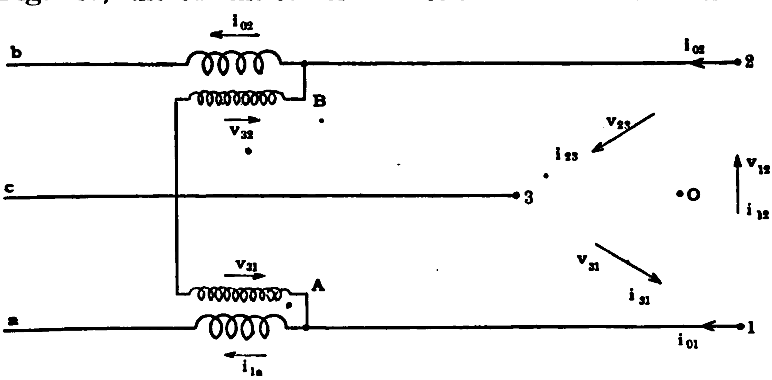


Fig. 135.

line wire a, and the current coil of B is connected in series with a second line wire b. The potential circuit of the meter A is connected across from a to the third line wire c, and the potential circuit of meter B is connected across from the line b to the line c. Let the instantaneous value of the currents and *p.d.'s* be represented by small letters with subscripts indicating their directions. The instantaneous torque on the moving element of meter A is then proportional to $v_{31}\, i_{01}$ and the instantaneous torque on the moving element of meter B in the *same relative direction* is $v_{32}\, i_{02}$. (That is, when the instantaneous currents and *p.d.'s* are actually in the directions indicated by the arrows on the coils, and the two meters are exactly alike, these are the values of the instantaneous torques in each meter either both to the left or both to the right.)

But
$$v_{32} = -v_{23}$$
$$i_{01} = i_{31} - i_{12}$$
$$i_{02} = i_{12} - i_{23}$$

Hence the total torque on the moving elements of both meters is
$$\begin{aligned} v_{31}\, i_{01} + v_{32}\, i_{02} &= v_{31}\ (i_{31} - i_{12}) - v_{23}\ (i_{12} - i_{23}) \\ &= v_{31}\, i_{31} + v_{23}\, i_{23} - (v_{31} + v_{23})\ i_{12} \end{aligned}$$

But
$$v_{31} + v_{23} = -v_{12}$$

(since the total drop of electric potential at any instant around a closed path is zero). Hence the total instantaneous torque on the moving elements of the two meters is proportional to
$$v_{12}\, i_{12} + v_{23}\, i_{23} + v_{31}\, i_{31}$$
which is equal to the total instantaneous power.

Since the reading of each wattmeter is proportional to the *average* torque acting on its moving element, it follows that the *algebraic* sum (the algebraic sum, since the average values of $v_{31}\, i_{01}$ and $v_{32}\, i_{02}$ may be of opposite sign) of the two wattmeter readings gives the true average power, independent of whether the load is balanced or not and also independent of the wave shape of current and *p.d.;* provided the wave shape of the current in the potential circuit is the same as the wave shape of the *p.d.* across it. This provision is only approximately fulfilled when the potential circuit of the meter contains iron.

Since the average value of the instantaneous products $v_{31}\, i_{01}$ and $v_{32}\, i_{02}$ may be either positive or negative, the deflection of the wattmeter needle may be either to the right or to the left of the zero.

Wattmeters, however, are usually designed with a scale reading only to the right; consequently when a positive value of the average of the product vi corresponds to a deflection to the right, then the needle goes off the scale to the left when the value of this product is negative. However, by reversing the leads connecting the current coil in the line, the direction of the current in the current coil is reversed with respect to the current in the potential coil, and the needle will then deflect to the right. (Note that the terminal of the potential circuit connected directly to the suspended coil of the meter should always be connected to the line wire in which the current coil of the meter is connected; otherwise, since the impedance of this coil is only a fraction of the total impedance of the potential circuit, practically the full voltage across the potential circuit will also exist between the potential coil and the current coil of the meter, and may cause the insulation between the two to break down.)

To measure, then, the total three-phase power by this two wattmeter method, it is necessary to connect the current coils of the two wattmeters in the lines a and b in such a manner that the needles of both meters deflect to the right. It is then necessary to determine whether the two readings shall be added or subtracted. This is determined from the fact that if the average value of the two products $v_{31}\, i_{01}$ and $v_{32}\, i_{02}$ are both of the same sign, then both meters will also read to the right if they are interchanged, and the connection to the middle wire c is kept unaltered. If, however, the average values of these two products are of the opposite sign, then when the two meters are interchanged and the connection to c is kept unaltered, the deflection of each instrument will reverse. Hence, the general rule, *subtract the two readings if on substituting one meter for the other (the connection to the common wire c being kept unaltered), the deflection reverses; otherwise the two readings are to be added.*

231. Two Wattmeter Method Applied to a Balanced Three-Phase System. — In case the load is balanced and the currents and *p.d.'s* are both harmonic functions or sine waves, the phase relations of the currents and potential drops in the two wattmeters can be readily deduced. Let θ be the angle by which the line current I_y *lags behind* the potential drop V_y (*i.e.*, $\cos\theta$ is the power factor of the load), then the vectors representing the various currents and *p.d.'s* are as shown in Fig. 136.

The average value of the instantaneous product $v_{31}\, i_{01}$ is then equal to the product of the lengths of the two vectors representing v_{31} and i_{01} by the cosine of the angle between them, that is

$$P_1 = \text{average } (v_{31}\, i_{01}) = V_\Delta\, I_y \cos\, (30° + \theta)$$

Similarly, the average value of the instantaneous product $v_{32}\, i_{02}$ is equal to the product of the lengths of the two vectors representing v_{32} and i_{02} by the cosine of the angle between them, that is

$$P_2 = \text{average } (v_{32}\, i_{02}) = V_\Delta I_y \cos\, (30° - \theta)$$

Hence when θ lies between $-60°$ and $+60°$ both P_1 and P_2 are positive and therefore their *sum* gives the true three-phase power; when θ is less than $-60°$ or greater than $+60°$, P_1 and P_2 are of opposite sign, and hence their *difference* gives the true three-phase power. But $\cos\, 60° = \frac{1}{2}$; hence when the power factor of

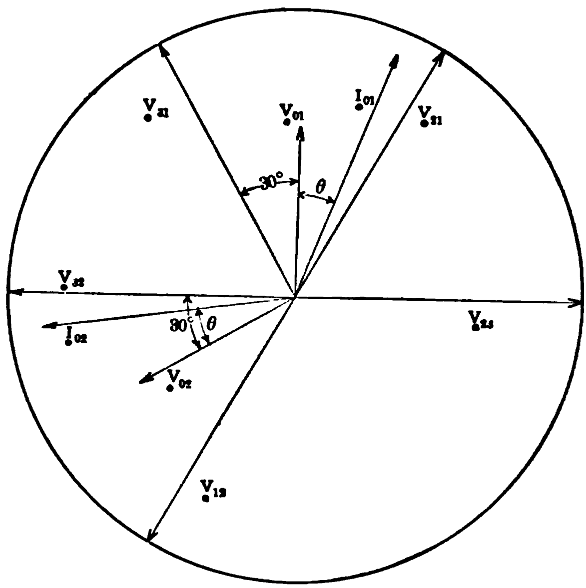


Fig. 136.

the load is greater than 50 per cent either leading or lagging, the sum of the wattmeter readings gives the true power; when the power factor of the load is less than 50 per cent the difference of the wattmeter readings gives the true power.*

In general, the power factor of the load is not known. How-

*Note that in the above discussion a negative value of θ means a leading current.

ever, in the case of a *balanced* load the following simple test may be used to determine whether the wattmeter readings are to be added or subtracted. Leaving all other corrections unchanged, transfer the connection of the potential coil of meter A, say, from the common wire c to the wire b. Then this meter will read the average value of the product $v_{21}\, i_{01}$, and therefore its reading will be

$$P_1' = V_\Delta\, I_y \cos(30° - \theta)$$

Hence when $-60 < \theta < +60$, P_1 and P_1' will both be of the same sign; when $\theta < -60$ or $\theta > +60$, P_1 and P_1' will be of the opposite sign. Consequently, if when this change in connection is made, the deflection of meter A remains in the same direction, the sum of the original readings of A and B gives the true power; if the deflection of A reverses, then the difference of the original readings of A and B gives the true power. An inspection of the vector diagram will show that this same rule applies to meter B when the connection of the potential coil of B to the common wire c is transferred to the wire a, all other connections remaining unaltered. Hence the general rule for a *balanced* load; connect the two meters in circuit in such a manner that both deflect to the right and take the two readings P_1 and P_2. Then, keeping all other connections unaltered, transfer the connection of the potential coil of one meter from the common wire to the wire in which the current coil of the second meter is connected; if the deflection of the former meter remains in the *same* direction *add* the original readings of the two meters; if the deflection of this meter reverses, subtract the original readings of the meters.

In the case of a balanced load the power-factor angle θ may be expressed directly in terms of the wattmeter readings. For, taking the sum and difference respectively of P_2 and P_1, we get

$$P_2 + P_1 = V_\Delta I_y \left[\cos(30° - \theta) + \cos(30° + \theta)\right] = \sqrt{3}\, V_\Delta I_y \cos\theta$$
$$P_2 - P_1 = V_\Delta I_y \left[\cos(30° - \theta) - \cos(30° + \theta)\right] = V_\Delta I_y \sin\theta$$

whence taking the ratio of these two expressions, we get

$$\tan\theta = \sqrt{3}\,\frac{P_2 - P_1}{P_2 + P_1}$$

SUMMARY OF IMPORTANT DEFINITIONS AND RELATIONS

1. The distance between the centers of successive magnetic poles of like sign in any electric machine is said to correspond to 360 **electrical degrees.**

2. A **two-phase generator** has two independent windings on its armature displaced 90 electrical with reference to each other; a **three-phase generator** has three independent windings on its armature, successive windings being spaced 120 electrical degrees with reference to each other.

3. A **Δ-connected** generator has its three windings connected in series and terminals brought out at the three junction points; a *Y*-connected generator has one end of each of its windings connected to a common junction point called the **neutral** and the other ends brought out to terminals.

4. The sum of the *e. m. f.'s* induced in the three windings of an alternator when these *e. m. f.'s* are sine waves is zero.

5. The **equivalent Δ e. m. f.** of a *Y*-connected alternator is numerically equal to the square root of three times the effective value of the *Y e. m. f.* and lags 90° behind the *e. m. f.* in the opposite branch of the *Y;* that is,

$$\dot{E}_{23} = -j\sqrt{3}\,\dot{E}_{01}$$

provided the *e. m. f.'s* are sine waves.

6. The **equivalent Y e. m. f.** of a Δ-connected alternator is numerically equal to the effective value of the Δ *e. m. f.* divided by the square root of three and leads by 90° the *e. m. f.* in the opposite side of the delta; that is,

$$E_{01} = j\frac{\dot{E}_{23}}{\sqrt{3}}$$

provided the *e. m. f.'s* are sine waves.

7. A three-phase system is said to be **balanced** when the currents and *e. m. f.'s* respectively in the three phases are equal in effective value and differ in phase by 120°.

8. The **equivalent Δ current** of a *Y*-connected alternator (or load) is numerically equal to the effective value of the *Y* current divided by the square root of three and lags 90° behind the current in the opposite branch of the *Y*; that is

$$\dot{I}_{23} = -j\frac{\dot{I}_{01}}{\sqrt{3}}$$

provided the currents and *e. m. f.'s* are sine waves and the system is balanced.

9. The **equivalent Y current** of a Δ-connected alternator (or load) is numerically equal to the square root of three times the effective value of the Δ current and leads by 90° the current in the opposite side of the Δ; that is,

$$\dot{I}_{01} = j\sqrt{3}\,\dot{I}_{23}$$

provided the currents and *e. m. f.'s* are sine waves and the system is balanced.

10. The **equivalent Δ impedance** of a *Y*-connected winding is equal to three times the *Y* impedance, that is,

$$\dot{Z}_{\Delta} = 3\,\dot{Z}_{y}$$

11. The **equivalent Y impedance** of a Δ-connected winding is equal to the Δ impedance divided by three, that is,

$$\dot{Z}y = \frac{1}{3}\,\dot{Z}_{\Delta}$$

provided the currents and *e. m. f.'s* are sine waves and the system is balanced.

12. When all parts of a **balanced three-phase system** have been reduced to equivalent *Y's*, each of the three phases may be **treated as a single phase circuit,** each such circuit being considered completed by a wire of zero impedance connecting all the neutrals, provided the currents and *e. m. f.'s* are sine waves.

13. The **total power** input into a three-phase load is

$$P = \sqrt{3}\,V\,I\cos\theta$$

where V is the *p.d.* between terminals (the Δ *p.d.*), I is the line current (the Y current) and θ is the power-factor angle (the difference in phase between the Y *p.d.* and Y current or between the Δ *p.d.* and Δ current). This relation holds only when the currents and *p.d.'s* are sine waves.

14. When **two wattmeters,** connected in a three-phase circuit as shown in Fig. 135, read P_1 and P_2 watts respectively, the total power is

$$P = P_1 \pm P_2$$

To determine which sign to use, substitute one meter for the other, keeping the connection to the common wire unchanged; if the deflection of this meter reverses take the difference, if the deflection does not reverse take the sum.

When the system is balanced and the currents and *e. m. f.'s* are sine waves, the power factor of the load is $\cos\theta$ where

$$\tan\theta=\sqrt{3}\,\frac{P_2-P_1}{P_2+P_1}$$

PROBLEMS

1. The ends of the three coils of a three-phase alternator are brought out to six slip rings. When the three coils on this alternator are connected in Y and the machine is operated at its rated speed and field excitation, the rated output of the machine is 30 kw. at 1000 volts. If the speed and field excitation are kept constant, what would be (1) the rated output of the machine, and (2) the rated voltage when the three coils are connected in Δ?

Ans.: (1) 30 kw.; (2) 577 volts.

2. A 25-cycle, three-phase, Δ-connected generator develops an *e. m. f.* per coil of 1000 volts effective value. This *e. m. f.* contains the third harmonic and the effective value of this harmonic is 10 per cent of the effective value of the resultant *e. m. f.* The resistance per coil of the generator is 0.1 ohm and the reactance per coil at 25 cycles is 0.3 ohm. (1) What will be the value of the current per coil when the generator is supplying no external load? (2) Will this current be a sine-wave current, and if so, what will be its frequency?

Ans.: (1) 110.4 amperes; (2) sine-wave current with frequency of 75 cycles.

3. A 500-volt, 25-cycle, 100-kilovolt-ampere, three-phase, Δ-connected alternator, while operating at full load on a circuit of 85% power factor, has one of its coils burnt out. With this coil open-circuited, (1) what is the maximum balanced load at 85% power factor which the machine can supply continuously? (2) What proportion of this load is supplied by each of the two remaining coils?

Ans.: (1) 49.1 kw.; (2) 32.1% and 67.9%.

4. The armature of a 230-volt, two-phase, 50-kilovolt-ampere alternator is re-wound so that the end of one of the coils, A, is connected to the middle of the other coil, B. If 13.4% of the turns on coil B are removed, what will be the rating of the resultant alternator?

Ans.: 230 volts, 3-phase, 43.3 kilovolt-amperes.

5. A coil of high impedance is connected between two slip-rings of a Y-connected 200-volt alternator. Find (1) the potential difference between the middle point of the impedence coil and the neutral point of the alternator, and (2) the potential difference between the middle point and the third terminal of the alternator.

Ans.: (1) 57.8 volts; (2) 173.2 volts.

6. The potential difference between each of the line wires A, B and C and the neutral 0 of a Y-connected load on a three-phase system is 100 volts and these respective Y voltages differ in phase by 120°. The impedances of the three branches of the load are respectively $\underset{\cdot}{Z}_{AO}=10+j\,0$, $\underset{\cdot}{Z}_{BO}=0+j\,10$ and $\underset{\cdot}{Z}_{CO}=0-j\,10$. (1) What will an ammeter connected between the neutral of the load and the neutral of the generator indicate? (2) If the current coils of two wattmeters are connected in lines A and B respectively and the potential coils are connected from A to C and from B to C respectively, what will each wattmeter read? (3) What is the total power supplied to the load?

Ans.: (1) 7.32 amperes; (2) 1500 watts and 866 watts; (3) 1000 watts.

7. A three-phase Y-connected generator supplies power to a balanced load at a line voltage of 220 volts. The power is measured by the two wattmeter method and each wattmeter reads 10 kw. If the resistance and reactance of each phase of the armature is 0.1 ohm and 0.3 ohm respectively, find the induced *e. m. f.* per phase of the alternator.

Ans.: 133.2 volts.

8. In series with each terminal of a balanced three-phase load is connected a non-inductive resistance of 0.5 ohm. The *p.d.* between the terminals of the load is 200 volts, the *p.d.* between line wires just outside the resistances is 300 volts, and the *p.d.* across the terminals of each resistance 100 volts. What is (1) the total power supplied to the load, and (2) its power factor?

Ans.: (1) 20 kw.; (2) 28.9%.

9. Each phase of a balanced three-phase, delta-connected load is formed by an impedance coil in series with a non-inductive resistance of 15 ohms. The voltage across the impedance coil in each phase is 80, the voltage across the non-inductive resistance in each phase is 150, the resultant voltage across the impedance coil and the non-inductive resistance is 200. What would be the

readings of the two wattmeters when connected to this load to measure the power by the two-wattmeter method?

Ans.: 3420 watts and 2190 watts.

10. The two wattmeters connected to a balanced three-phase load read 5240 watts and 2380 watts respectively. (1) What is the difference in phase between the current in the current coil of the first wattmeter and the potential drop through the potential coil of the second wattmeter? (2) What is the power factor of the load?

Ans.: (1) 123 degrees or 57 degrees; (2) 83.9%.

11. The current coil of a single wattmeter is connected in one of the line wires of a balanced three-phase load which takes 30 kilowatts at a lagging power factor of 80%. Call this wire A and the other two line wires B and C respectively. What would the wattmeter read when its potential coil is connected (1) between A and B, (2) between A and C, and (3) between B and C?

Ans.: (1) 21.5 kilowatts; (2) 8.5 kilowatts; and (3) 13.0 kilowatts.

12. 20,000 kilowatts is supplied from a generating station to a substation 50 miles distant over a three-phase line formed of three No. 0000 B. & S. copper wires spaced 6 feet between centers. The resistance of each wire per mile is 0.258 ohms and the inductance per mile of each wire is 1.93 millihenries. A voltage of 60,000 is maintained between wires at the substation and the power factor of the substation is 80% lagging current. The system is balanced. Calculate (1) the voltage between wires at the generating station, (2) the total number of kilowatts lost in the transmission line and (3) the power factor at the generating station.

Ans.: (1) 68,200 volts; (2) 2250 kilowatts; (3) 78.3%.

13. Energy is supplied from a three-phase Y-connected generator over a three-phase line to a balanced three-phase load formed by two motors connected in parallel. The motors take respectively 50 kilowatts at 80% power factor lagging and 100 kilowatts at 90% power factor leading. The potential difference between the terminals of the motors is 500 volts and the resistance and reactance of each line wire are respectively 0.1 ohm and 0.2 ohm. (1) What is the symbolic expression for the total current per wire taken by the load referred to the potential drop between the terminal of one terminal of the load and the neutral?

(2) What is the numerical value of the potential difference between the generator terminal and the neutral?

Ans.: (1) $173.3+j0.3$; (2) 308 volts.

14. A three-phase Y-connected generator delivers power to a balanced delta-connected load over a transmission line. The resistance of each line wire is 0.3 ohm and the reactance of each line wire is 0.5 ohm. The potential differences at the terminals of the generator and the load are respectively 500 volts and 400 volts. The power factor of the load is 80% lagging. Calculate (1) the line current, (2) the power delivered to the load, (3) the power output of the generator, (4) the power factor of the generator and (5) the efficiency of the transmission line at this load.

Ans.: (1) 104.7 amperes; (2) 58.0 kilowatts; (3) 67.9 kilowatts; (4) 78.4%; (5) 85.4%.

APPENDIX A

THE following abbreviations are recommended by the American Institute of Electrical Engineers and are used in all its publications. In general these abbreviations should be used only when expressing definite numerical values.

NAME	INSTITUTE STYLE
Alternating current	spell out, or a-c. when used as compound adjective
Amperes	spell out
Brake horse power	b.h.p.
Boiler horse power	Boiler h.p.
British thermal units	B.t.u.
Candle-power	c-p.
Centigrade	cent.
Centimetres	cm.
Circular mils	cir. mils
Counter electromotive force	counter e.m.f.
Cubic	cu.
Diameter	spell out
Direct current	spell out, or d-c. when used as compound adjective
Electric horse power	e.h.p.
Electromotive force	e.m.f.
Fahrenheit	fahr.
Feet	ft.
Foot-pounds	ft-lb.
Gallons	gal.
Grains	gr.
Grammes	g.
Gramme-calories	g-cal.
High-pressure cylinder	spell out
Hours	hr.
Inches	in.
Indicated horse power	i.h.p.
Kilogrammes	kg.
Kilogramme-metres	kg-m.
Kilogramme-calories	kg-cal.
Kilometres	km.
Kilowatts	kw.
Kilowatt-hours	kw-hr.
Magnetomotive force	m.m.f.
Mean effective pressure	spell out
Miles	spell out

Name	Institute Style
Miles per hour per second	Miles per hr. per sec.
Millimetres	mm.
Milligrammes	mg.
Minutes	min.
Metres	m.
Metre-kilogrammes	m-kg.
Microfarad	spell out
Ohms	spell out
Per	spell out
Percentage	per cent, or % in tabular matter only
Pounds	lb.
Power-factor	spell out
Revolutions per minute	rev. per min., or r.p.m.
Seconds	sec.
Square	sq.
Square-root-of-mean-square	effective, or r.m.s.
Ton-mile	spell out
Tons	spell out
Volts	spell out
Volt-amperes	spell out
Kilovolts	kv.
Kilovolt-amperes	kv-a.
Watts	spell out
Watt-hours	watt-hr.
Watts per candle-power	watts per c-p.
Yards	yd.

1. Use lower case characters for abbreviations. An exception to this rule may be made in the case of words spelled normally with a capital. Example: "B.t.u." and not "b.t.u." nor "B.T.U." (British thermal unit). "U.S.gal." (United States gallon); "B. & S. gauge" (Brown & Sharpe gauge).

2. Use all abbreviations in the singular. Example: "17 lb." and not "17 lbs." (17 pounds). "14 in." not "14 ins." (14 inches).

3. Use a hyphen to connect abbreviations in cases where the words would take a hyphen if written out in full. When a hyphen is used, omit the period immediately preceding the hyphen. Example: "3 kw-hr." and not "3 kw.-hr." (3 kilowatt-hours).

4. Use a period after each abbreviation. In a compound abbreviation, do not use a space after the period. Example: "i.h.p." and not "i. h. p." (indicated horse power).

5. Never use "P." for "per" but spell out the word. Example: "100 ft-lb. per ton" (100 foot-pounds per ton); "60 miles per hr." (60 miles per hour).

6. Use "Fig." not "Figure." Example: "Fig. 3" and not "Figure 3."

7. In all decimal numbers having no units, a cipher should be placed before the decimal point. Example: "0.32 lb." not ".32 lb."

8. Use the word "by" instead of "x" in giving dimensions. Example: "8 by 12 in." not "8 x 12 in."

9. Never use the characters (′) or (″) to indicate either feet and inches, or minutes and seconds as period of time.

10. Use capitals sparingly; when used as units, do not capitalize volt, ampere, watt, farad, henry, ohm, coulomb, etc.

11. Do not use the expression "rotary" or "rotary converter"; use "converter" or "synchronous converter."

12. Do not use a descriptive adjective as a synonym for the noun described. Example: a "spare transformer," not a "spare"; a "portable instrument," not a "portable"; "automatic apparatus," not "automatics"; a "short circuit," not a "short."

13. Do not use the expressions a.c. current or d.c. current; a.c. voltage and d.c. voltage. Their equivalents, "alternating-current current," "direct-current current," "alternating-current voltage," "direct-current voltage" are absurdities.

14. Do not use the expressions, raising transformer," "lowering transformer"; these expressions are ambiguous. Use "step-up transformer," "step-down transformer."

15. Do not use the words "primary" and "secondary" in connection with transformer windings. Use instead "high-tension" and "low-tension."

NOTATION.

The following notation forms ¶324 of Section V of the Standardization Rules of the Institute.

E, e, voltage, e.m.f., potential difference

I, i, current	Z, z, impedance
P, power	L, l, inductance
ϕ, magnetic flux	C, c, capacity
β, B, magnetic density	Y, y, admittance
R, r, resistance	b, susceptance
x, reactance	G, g, conductance

APPENDIX B

In this country wires for electrical purposes, when less than half an inch in diameter, are nearly always specified in terms of a wire gauge introduced by the Brown and Sharpe Manufacturing Co. This gauge, called briefly the B. & S. gauge,* is such that successive sizes differ in diameter by a constant percentage. A solid wire 460 mils in diameter is called a No. 0000 wire and a wire 5 mils in diameter is called a No. 36 wire. The next smaller size to a No. 0000 wire is No. 000, the next smaller size No. 00, the next No. 0, the next No. 1 and so on up to No. 36. The ratio of the diameters of No. 0000 and No. 36 is $\frac{460}{5}=92$, and the ratio of the diameters of successive sizes is constant; this constant is therefore equal to the 39th root of 92. The 39th root of 92 is approximately equal to the sixth root of 2; hence the following approximate relations (since the cross section varies as the square of the diameter and the cube root of 2 is approximately 1.26):

1. The ratio of the cross sections of wires of successive sizes on the B. & S. gauge is equal to 1.26, the larger number on the gauge corresponding to the smaller cross section. This same relation holds for the weights of successive sizes for a given length.
2. The ratio of the resistances of wires of successive sizes on the B. & S. gauge is equal to 1.26, the larger number on the gauge corresponding to the larger resistance.
3. An increase of 3 in the gauge number halves the cross section and weight and doubles the resistance.
4. An increase of 10 in the gauge number decreases the cross section and weight to one-tenth their original values and increases the resistance to 10 times its original value.

The cross section of a No. 10 wire is approximately 10,000 circular mils. The resistance of this size copper wire is approximately 1 ohm per 1000 feet at 20° cent. and its weight approximately 31.5 pounds per 1000 feet. From the above relations the resistance, cross section and weight of any size of wire may be calculated approximately with but little effort. The resistance of a No. 10 aluminum wire is approximately 1.6 ohms per 1000 feet at 20° cent. and its weight approximately 15 pounds per 1000 feet.

The above relations are for solid wire. Stranded wire is numbered on the B. &. S gauge in accordance with its cross section, not its diameter. The diameter of a concentric strand is approximately 15% greater than that of a solid wire of the same number; its weight and resistance are from 1 to 2 per cent greater, depending upon the number of twists per unit length and the number of wires in the strand.

Below are given the exact relations between gauge numbers, diameter, cross section, weight and resistance for copper wire of 100% conductivity Matthiessen's Standard. For aluminum wire of 62% conductivity multiply the weights by 0.47 and the resistances by 1.613.

* This gauge is also called the American Wire Gauge, abbreviated A. W. G.

Solid Copper Wire — 100% Matthiessen's Standard

No. B.& S.	Diam. Mils	Area Cir. Mils	Weight, Pounds			Resistance, 20° C. 68° F.	
	Bare		1000′	Mile	Feet per lb.	1000′	Mile
0000	460	211,600	640.5	3,381	1.561	0.04913	0.2594
000	409.6	167,800	508	2,682	1.969	0.06195	0.3271
00	364.8	133,100	402.8	2,127	2.482	0.07811	0.4124
0	324.9	105,500	319.5	1,687	3.130	0.09850	0.5210
1	289.3	83,690	253.3	1,337	3.947	0.1242	0.6557
2	257.6	66,370	200.9	1,062	4.977	0.1566	0.8270
3	229.4	52,630	159.3	841.1	6.276	0.1975	1.043
4	204.3	41,740	126.4	667.4	7.914	0.2490	1.315
5	181.9	33,100	100.2	529.0	9.980	0.3141	1.658
6	162.0	26,250	79.46	419.5	12.580	0.3960	2.091
7	144.3	20,820	63.02	332.7	15.87	0.4993	2.636
8	128.5	16,510	49.98	263.9	20.01	0.6296	3.324
9	114.4	13,090	39.63	209.2	25.23	0.7940	4.192
10	101.9	10,380	31.43	166.0	31.82	1.001	5.286
11	90.74	8,234	24.93	131.6	40.12	1.262	6.664
12	80.81	6,530	19.77	104.4	50.59	1.592	8.407
13	71.96	5,178	15.68	82.79	63.79	2.008	10.60
14	64.08	4,107	12.43	65.63	80.44	2.531	13.36
15	57.07	3,257	9.858	52.05	101.4	3.192	16.85
16	50.82	2,583	7.818	41.28	127.9	4.025	21.25
17	45.26	2,048	6.200	32.74	161.3	5.075	26.80
18	40.30	1,624	4.917	25.96	203.4	6.399	33.79
19	35.89	1,288	3.899	20.59	256.5	8.070	42.61
20	31.96	1,022	3.092	16.33	323.4	10.18	53.75

Stranded Copper Wire — 100% Matthiessen's Standard

No. B. & S.	Diam. Mils Bare	Area Cir. Mils	Weight, Pounds 1000′	Weight, Pounds Mile	Weight, Pounds Feet per lb.	Resistance, 20° C. 68° F. 1000′	Resistance, 20° C. 68° F. Mile
		2,000,000	6,100	32,210	0.164	0.005198	0.02744
		1,500,000	4,575	24,160	0.219	0.006930	0.03659
		1,250,000	3,813	20,130	0 262	0.008316	0.04391
	1,152	1,000,000	3,050	16,100	0.328	0 01040	0.05489
	1,125	990,000	2,898	15,300	0.345	0.01094	0.05778
	1,092	900,000	2,745	14,490	0.364	0.01155	0.06099
	1,062	850,000	2,593	13,690	0.385	0.01223	0.06458
	1,035	800,000	2,440	12,880	0.409	0.01299	0.06861
	959	750,000	2,288	12,080	0.437	0.01386	0.07318
	963	700,000	2,135	11,270	0.468	0.01485	0.07841
	927	650,000	1,983	10,470	0.504	0 01599	0.08444
	891	600.000	1,830	9,662	0.546	0.01732	0.09145
	855	550,000	1,678	8,857	0.596	0.01890	0.09980
	819	500.000	1,525	8,052	0.655	0.02079	0.1098
	770	450,000	1,373	7,247	0.728	0.02310	0.1220
	728	400,000	1,220	6,442	0.819	0.02599	0.1372
	679	350,000	1,068	5 636	0.936	0.02970	0.1568
	630	300,000	915	4,831	1.093	0.03465	0.1830
	590	250,000	762	4,026	1.312	0.04158	0.2196
0000	530	211,600	645	3,405	1.550	0.04913	0.2594
000	470	167,800	513	2,709	1.949	0.06195	0.3271
00	420	133,100	406	2,144	2.463	0.07811	0.4124
0	375	105,500	322	1,700	3.106	0.09850	0.5210
1	330	83,690	255	1,347	3.941	0.1242	0.6557
2	291	66,370	203	1,072	4.926	0.1566	0.8270
3	261	52,630	160	845	6.250	0.1975	1.043
4	231	41,470	127	671	7.874	0.2490	1.315

This table is calculated for untwisted strands; if the strand is twisted the cross section of the copper at right angles to the length of the strand, the weight per unit length and the resistance per unit length will each increase from 1 to 3 per cent, and the length per unit weight will decrease from 1 to 3 per cent, depending on the number of twists per unit length and the number of wires in the strand.

INDEX

Summaries of important definitions and principles will be found on the following pages: